An introduction to Topological Quantum Computing

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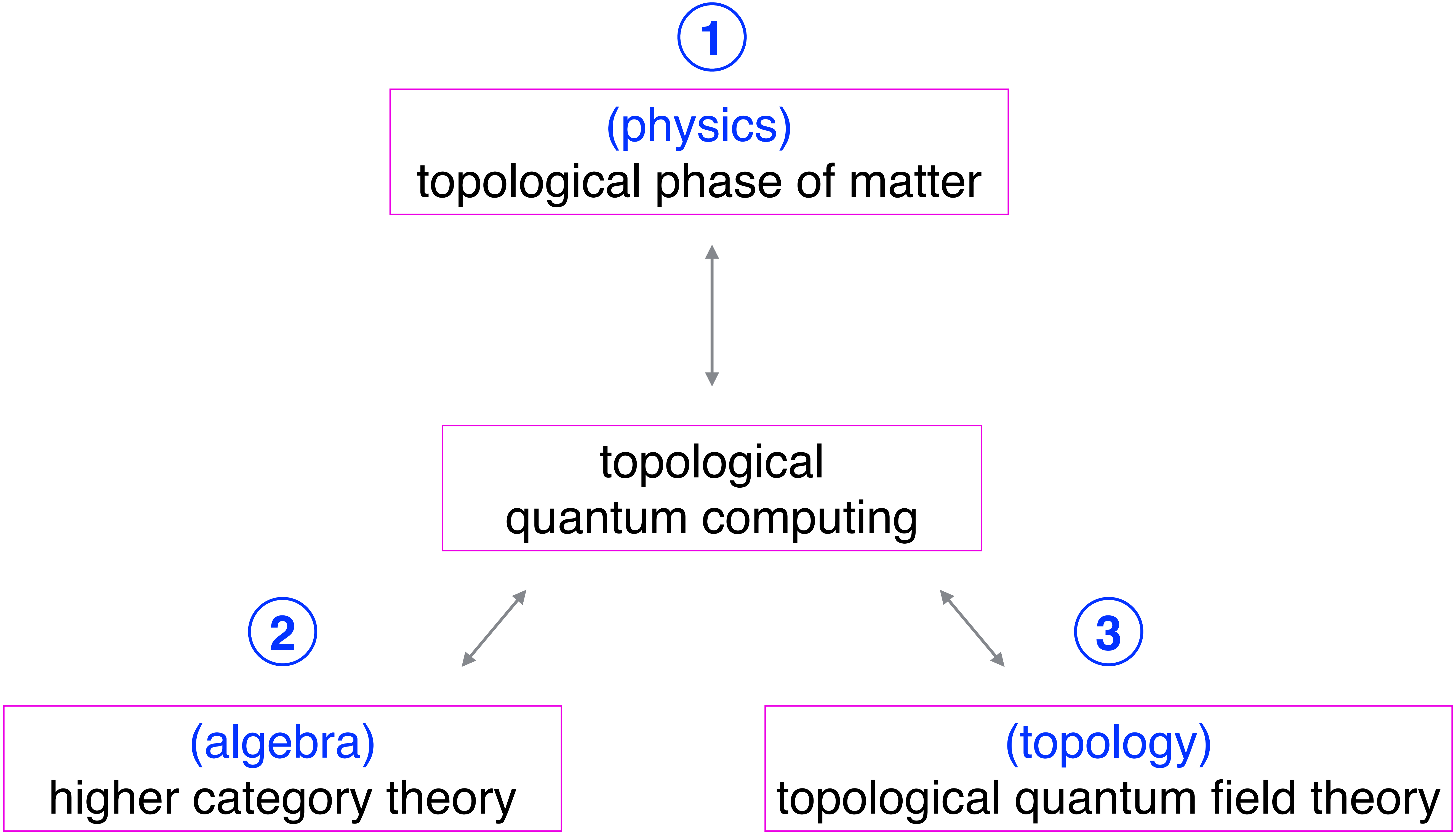
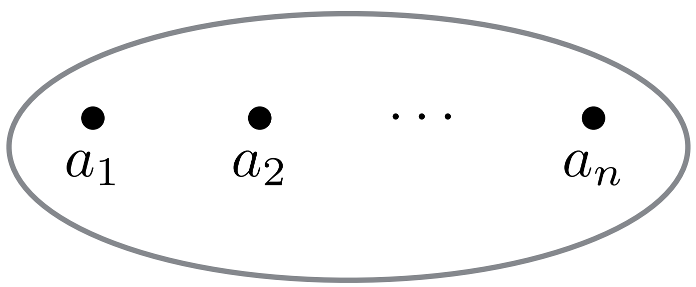
Mini course for
Algebraic structures in topology 2026

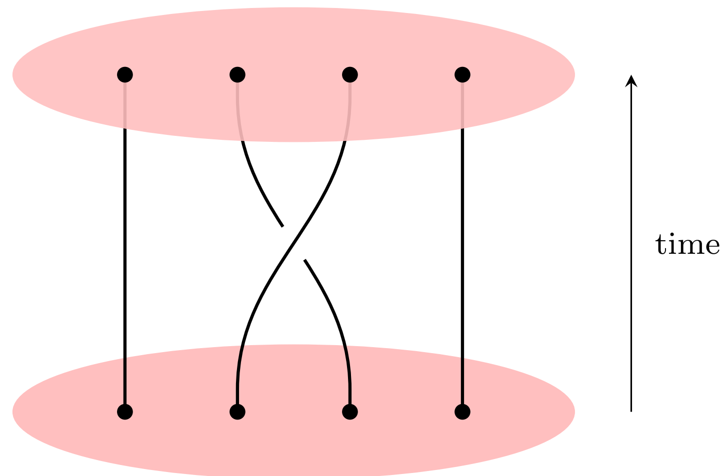
University of Puerto Rico, Río Piedras, July 10-17, 2026

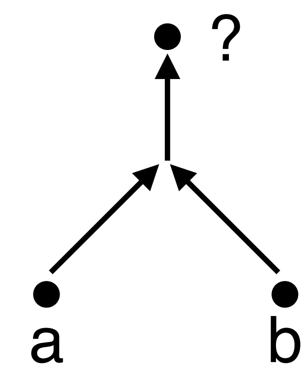

1
(physics)
topological phase of matter
topological
quantum computing
2
(algebra)
higher category theory
3
(topology)
topological quantum field theory

Recall:

Anyons can be created, braided, and fused.







Goal:

Describe the theory of anyons in “completely” elementary ways with physics motivation.

Algebraic Description of Anyons

An anyon system is characterized by the following data.

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Example: Fibonacci theory Fib

$$L = \{\mathbf{1}, \tau\}, \quad \bar{\tau} = \tau$$

★ Fusion rules

Fusing anyons a and b may result in different anyons c, d, …

Define $N_{ab}^{c} = 1$ if c is a possible outcome, and 0 otherwise.

Remark: In general, N_{ab}^{c} can be > 1.

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$$a \otimes b = b \otimes a \qquad \mathbf{1} \otimes b = b \qquad (a \otimes b) \otimes c = a \otimes (b \otimes c)$$

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Example: Fib $\qquad \tau \otimes \tau = \mathbf{1} \oplus \tau$

★ State spaces

For a collection of anyons $c, a_1, a_2, \cdots, a_n$, there are two Hilbert spaces

$V_c^{a_1,\cdots,a_n}$: space of physical processes of splitting c into $a_1, \cdots, a_n$

$V_{a_1,\cdots,a_n}^c$: space of physical processes of fusing $a_1, \cdots, a_n$ into c.

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There is a `time-reverse'/adjoint operator:

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$n = 1$:

$$V_c^a = V_a^c = \delta_{a,c}\mathbb{C} = \delta_{a,c}\ \mathrm{span}\{\ \Big|\, c\ \}$$

`identity' process

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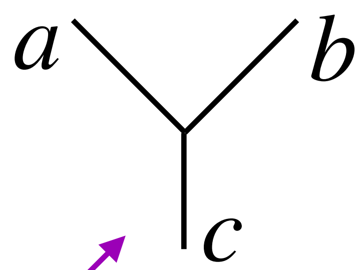
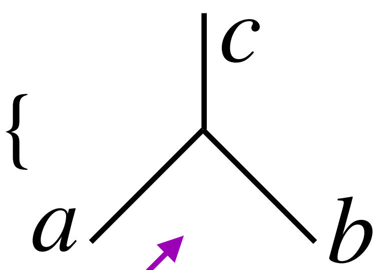
$V_c^{ab} = \mathrm{span}\{$  $\}$

splitting process

$V_{ab}^c = \mathrm{span}\{$  $\}$

fusion process

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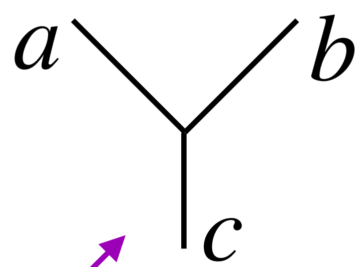
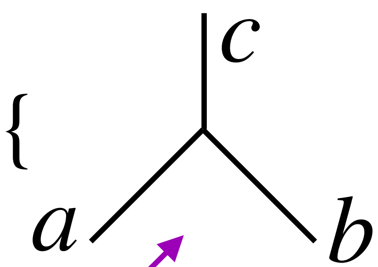
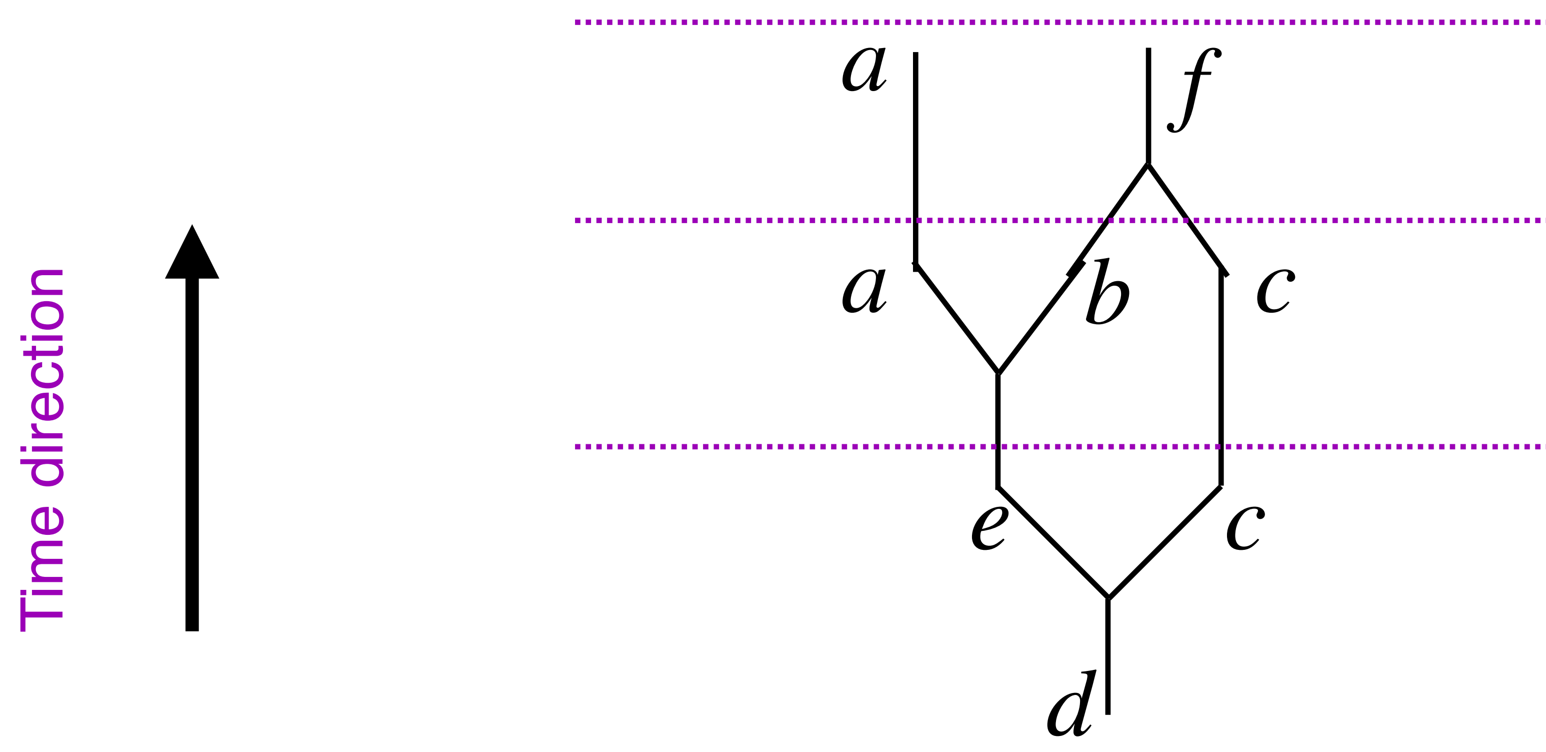
$V_c^{ab} = \mathrm{span}\{$  $\}$

$V_{ab}^c = \mathrm{span}\{$  $\}$

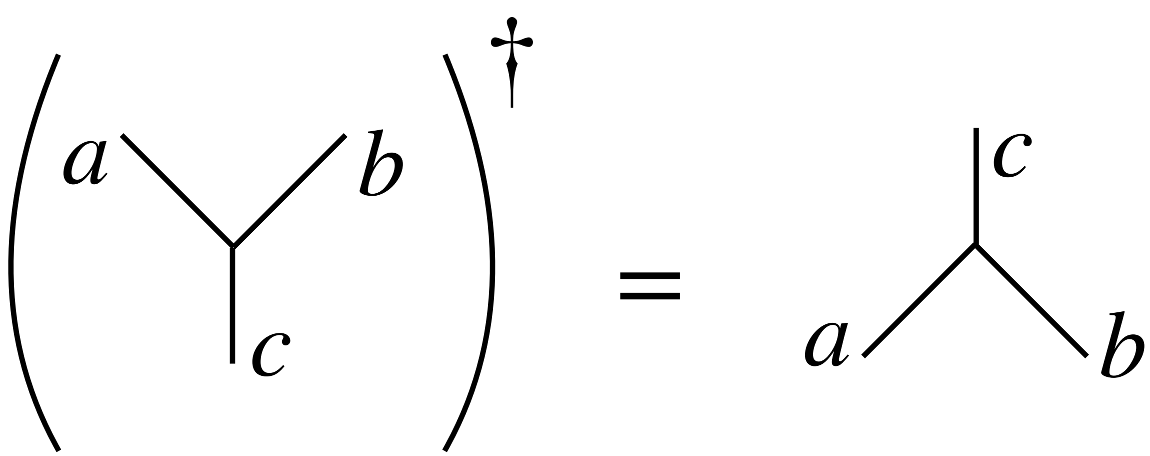
splitting process

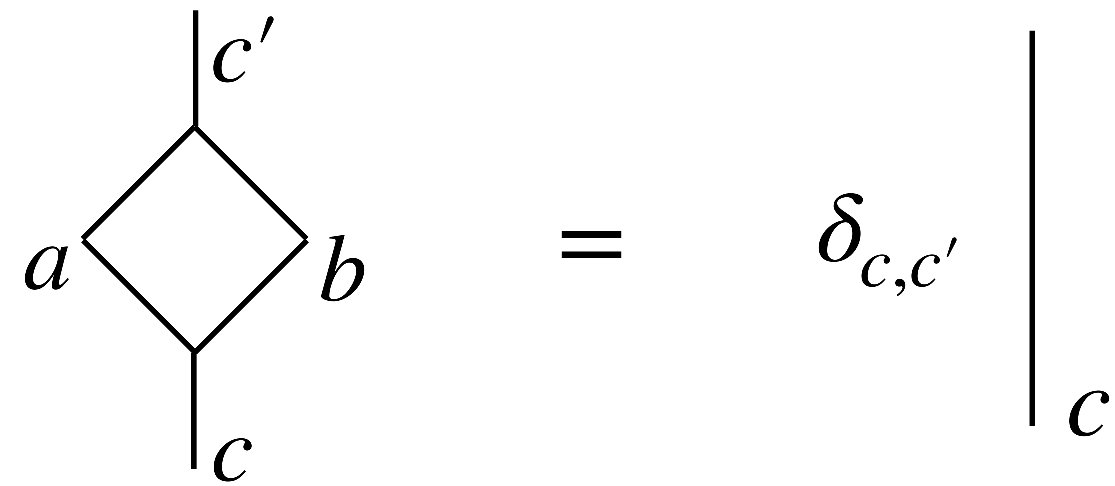
fusion process

Processes can be “composed” and happen “in parallel”:



Make the choice so that,





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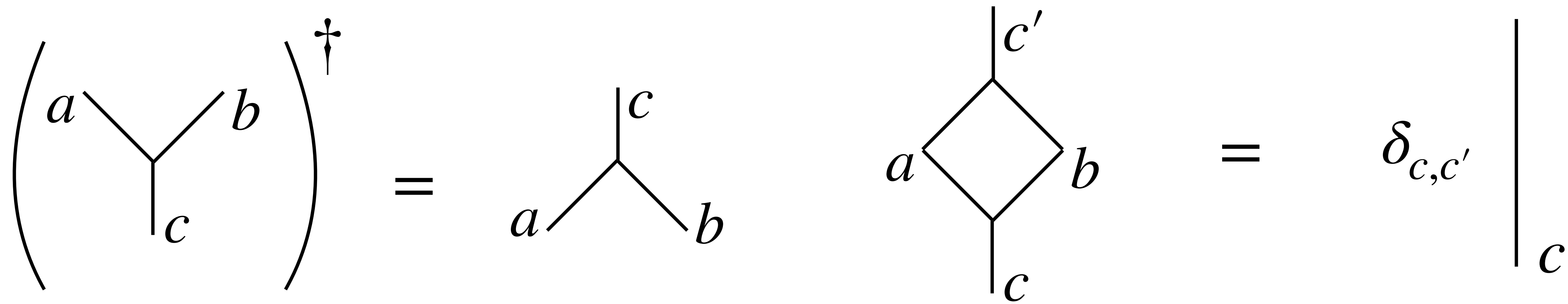
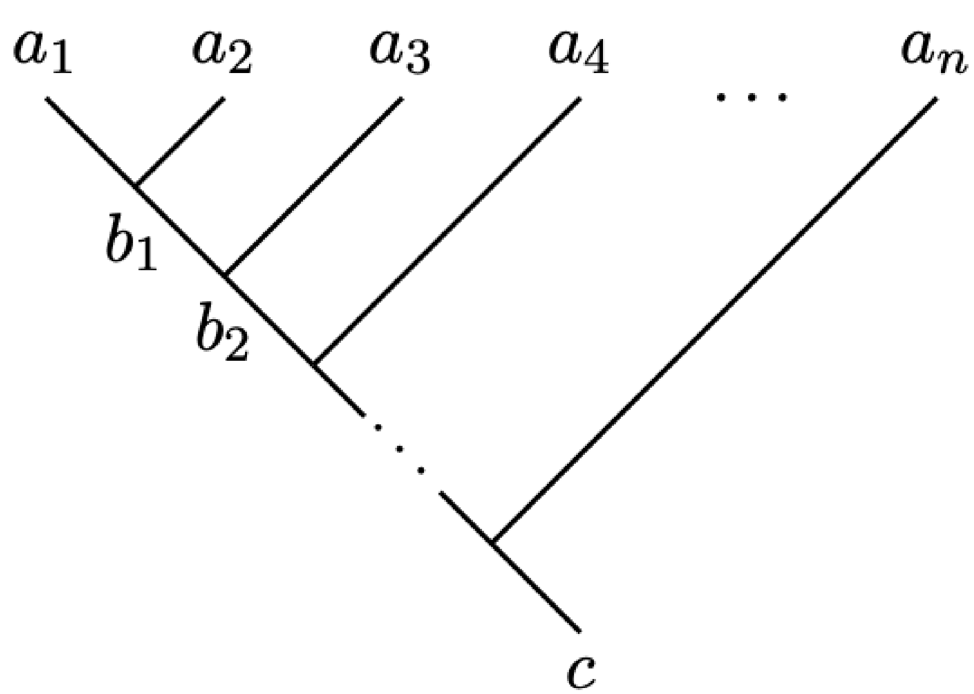
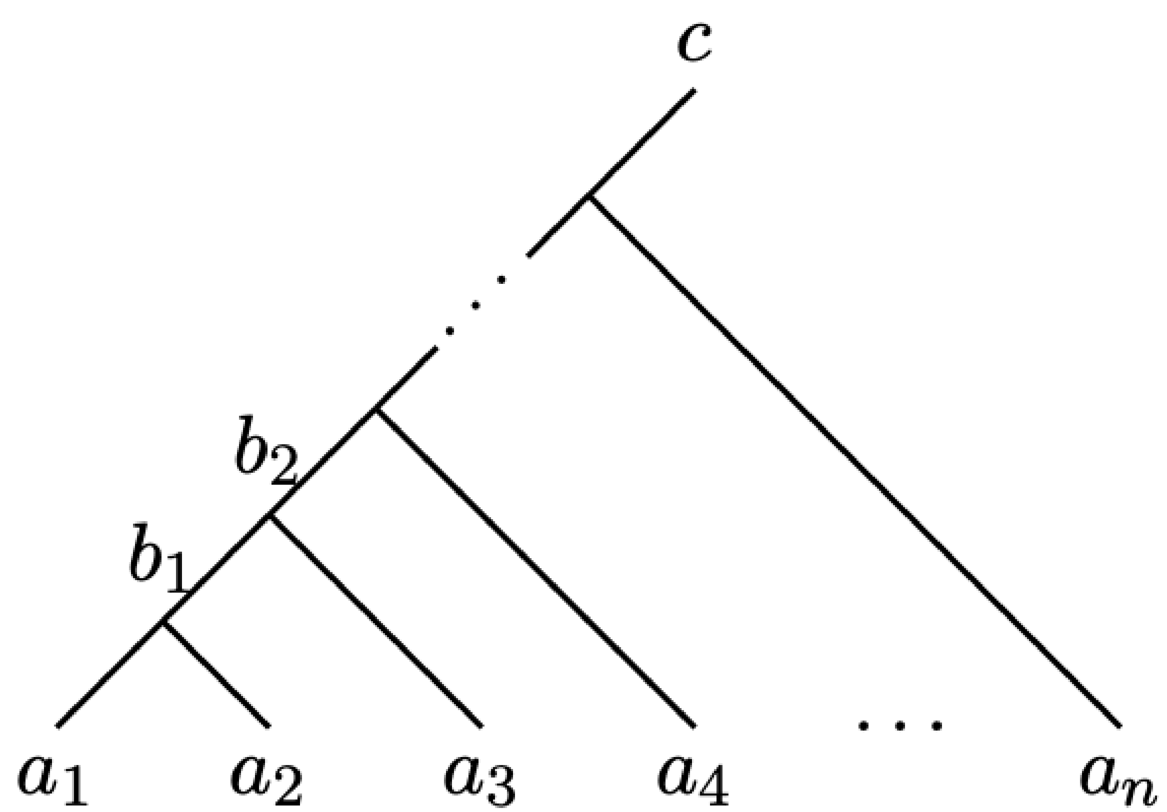


For general n, an orthonormal basis of $V_c^{a_1,\cdots,a_n}$ is given by fixing a binary tree and varying labels of internal edges:



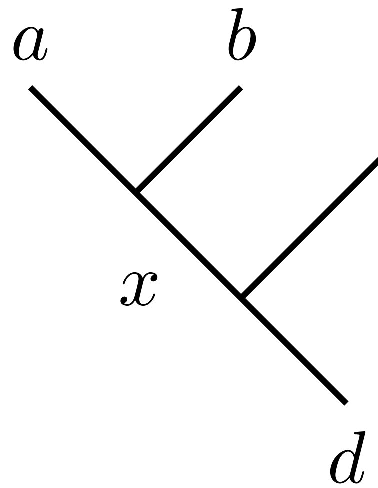
splitting c into b_{n-2} and a_n, followed by splitting b_{n-2} into b_{n-3} and a_{n-1}, ……, followed by splitting b_1 into a_1 and a_2

The adjoint gives an orthonormal basis of $V^{c}_{a_1,\cdots,a_n}$:

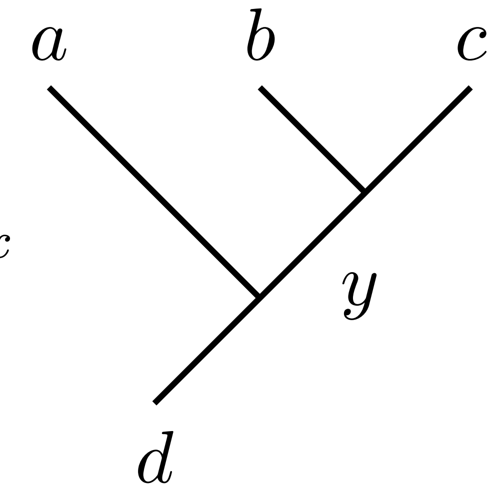


fusing a_1 and a_2 into b_1, followed by
fusing b_1 and a_3 into b_2, ……

Relations between different bases:



$$= \sum_y F^{abc}_{d;yx}$$



F-move relating two bases of V^{abc}_d

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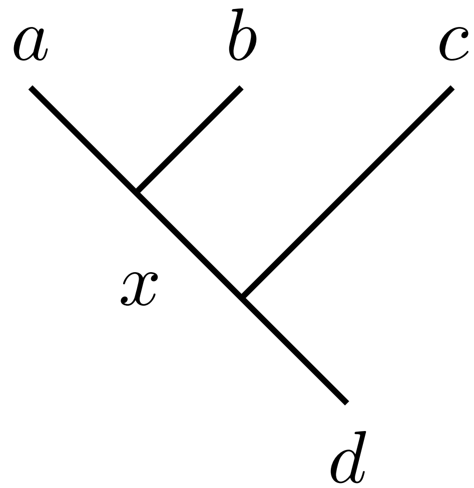

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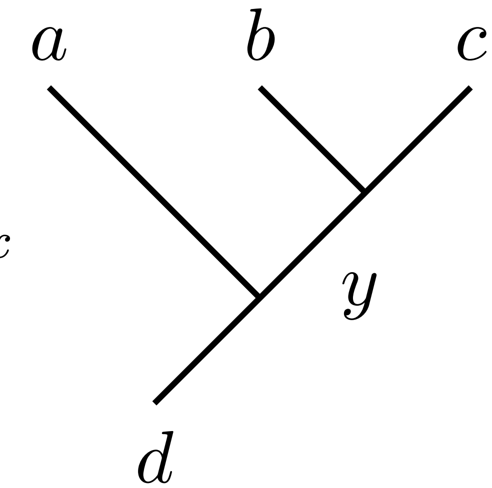

F-move relating two bases of V_d^{abc}

Possible x-values $x_1, \cdots, x_k$; possible y-values $y_1, \cdots, y_k$. Define the matrix F_d^{abc} with $(F_d^{abc})_{ij} = F^{abc}_{d;y_i x_j}$. It is unitary.

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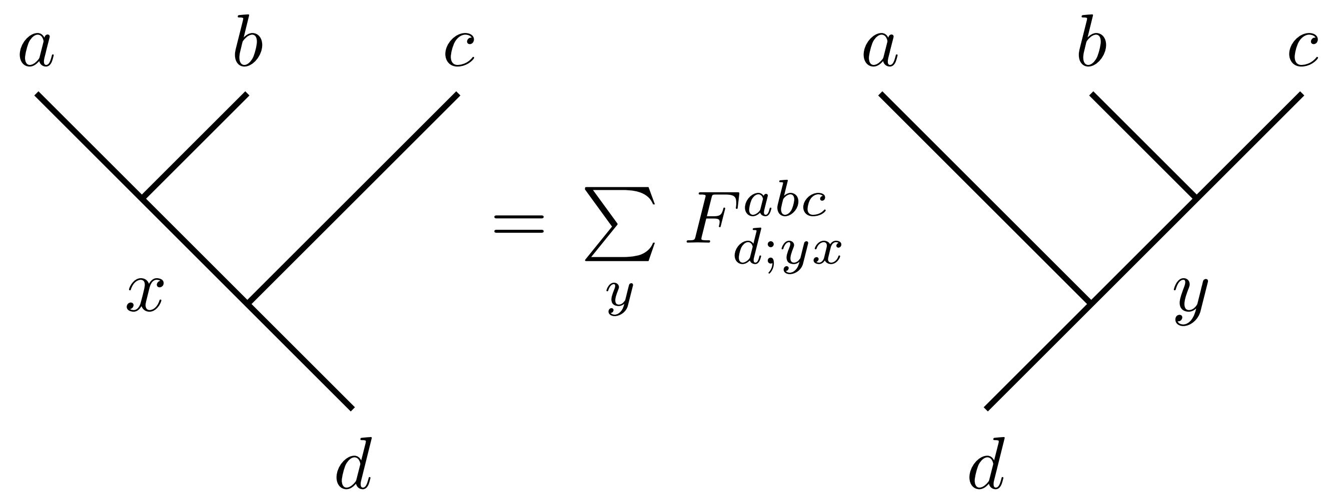

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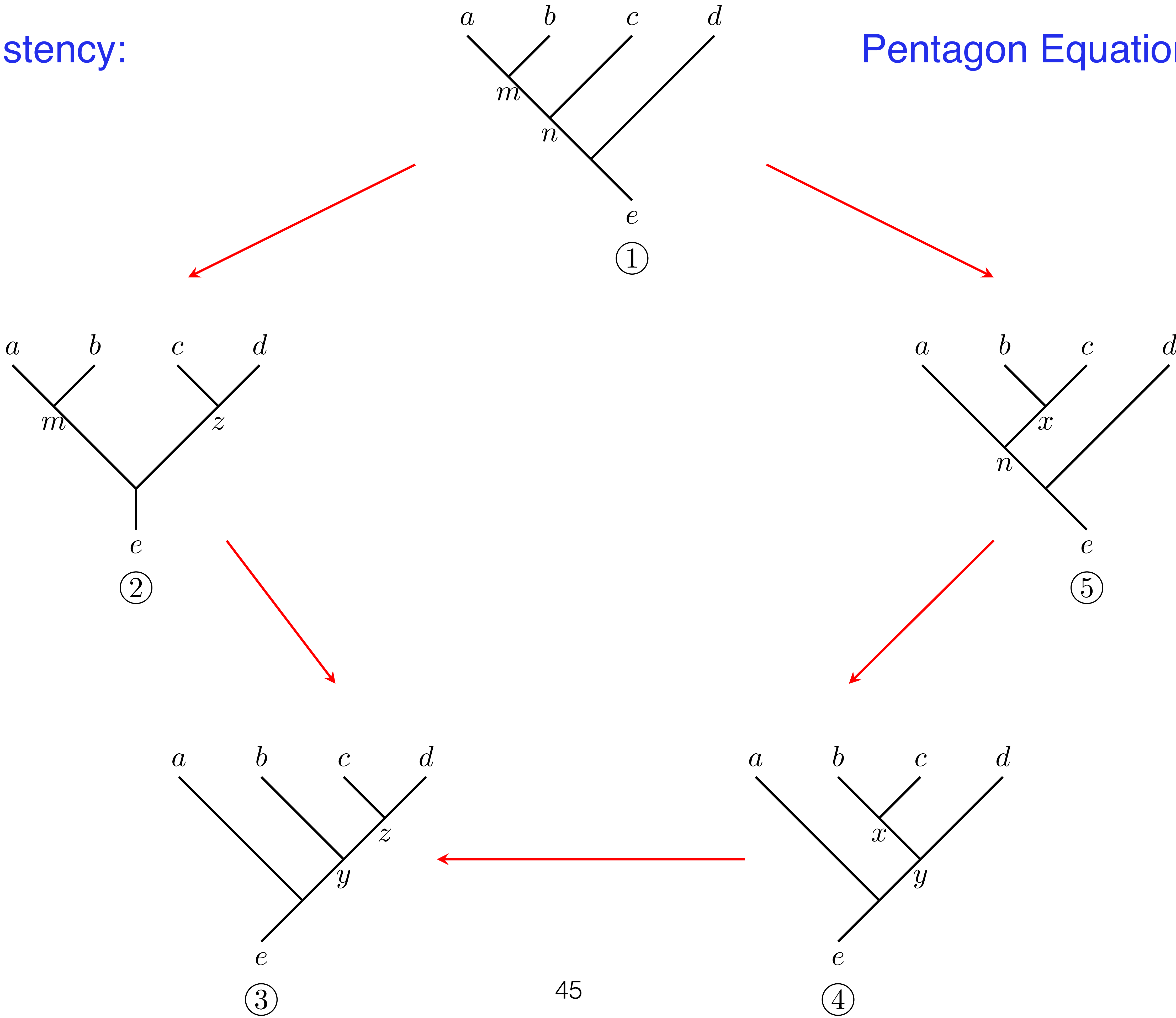
Example: Fib

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \qquad \phi = \frac{1+\sqrt{5}}{2}$$

Consistency: Pentagon Equation



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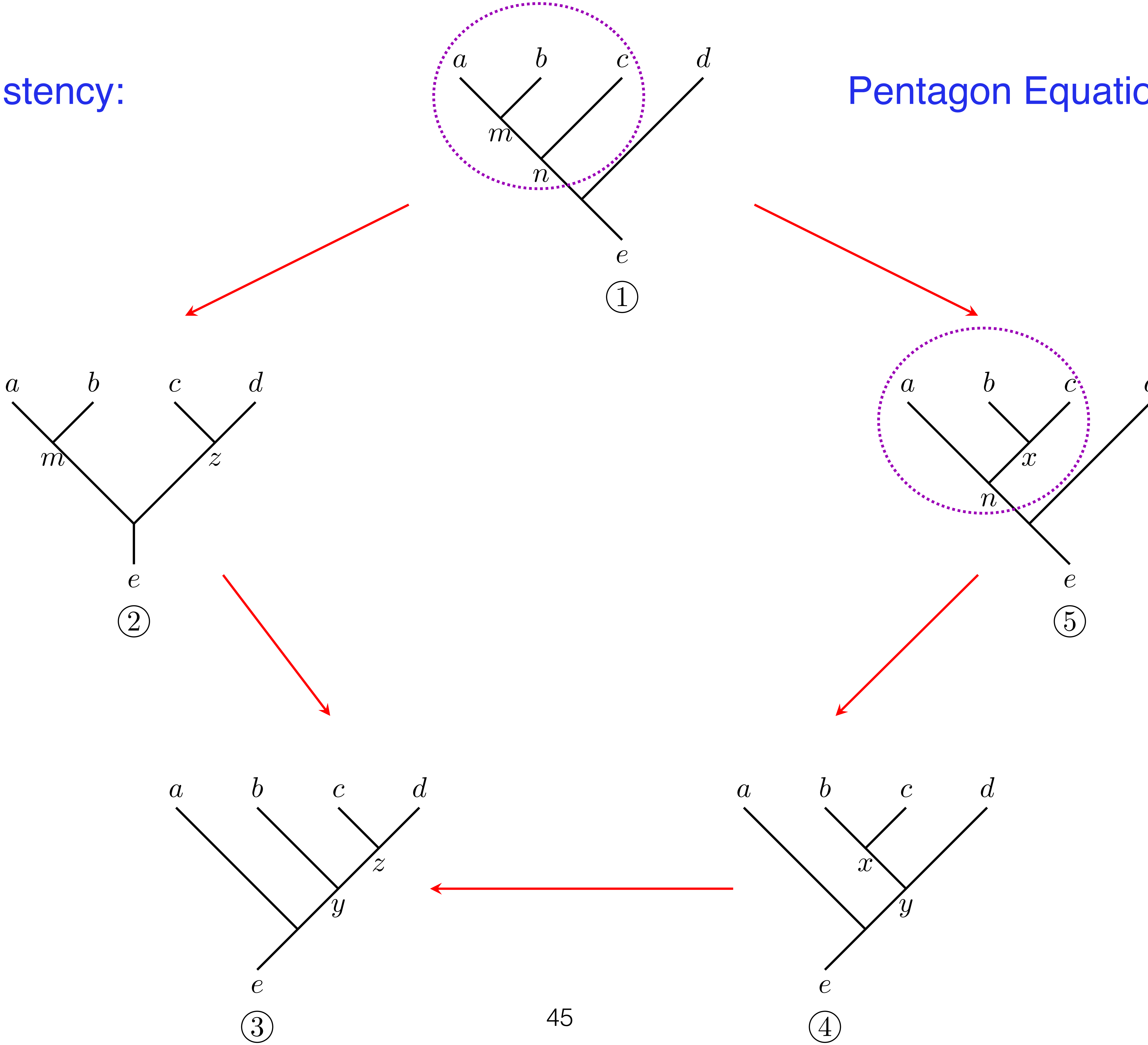

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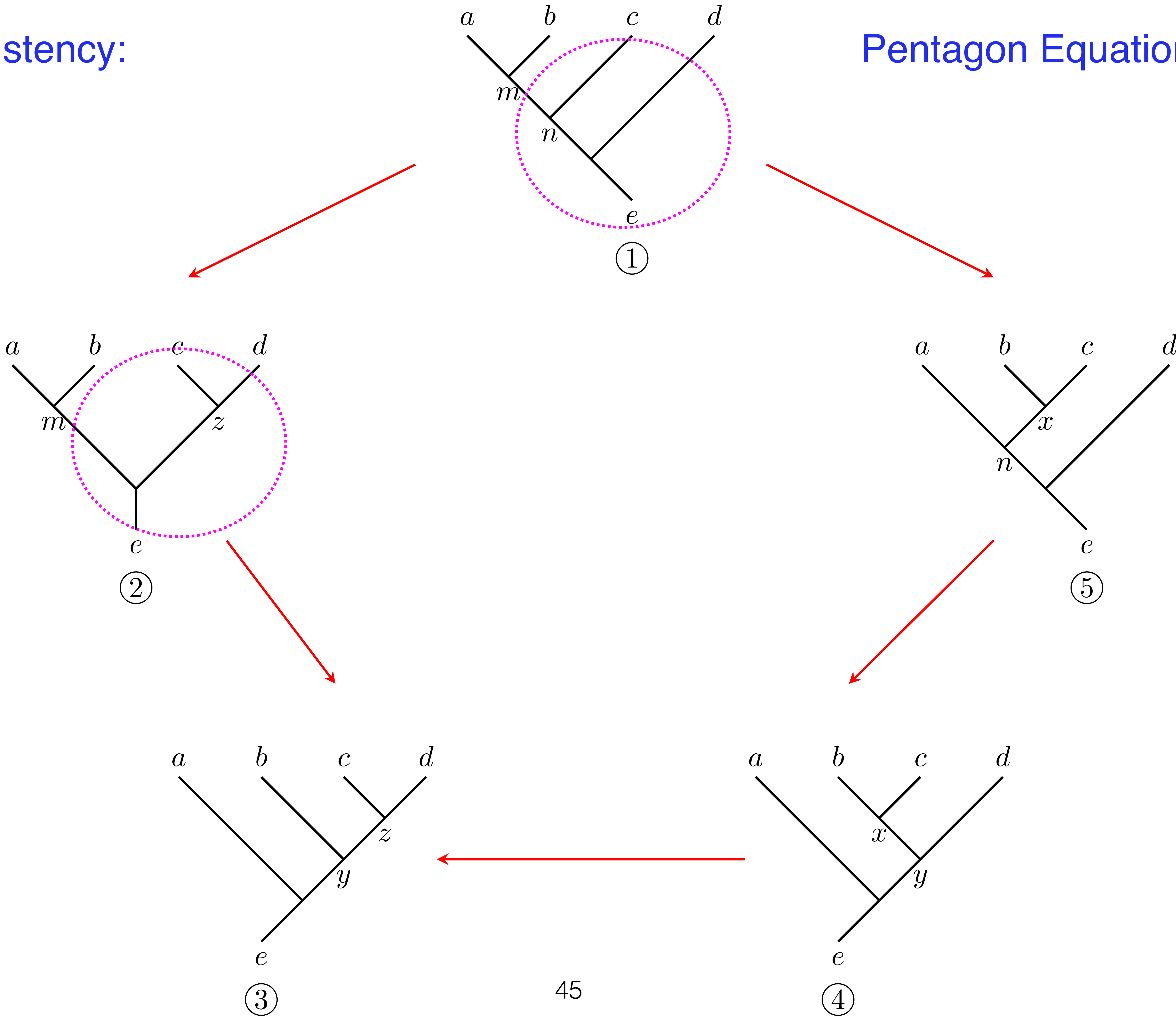
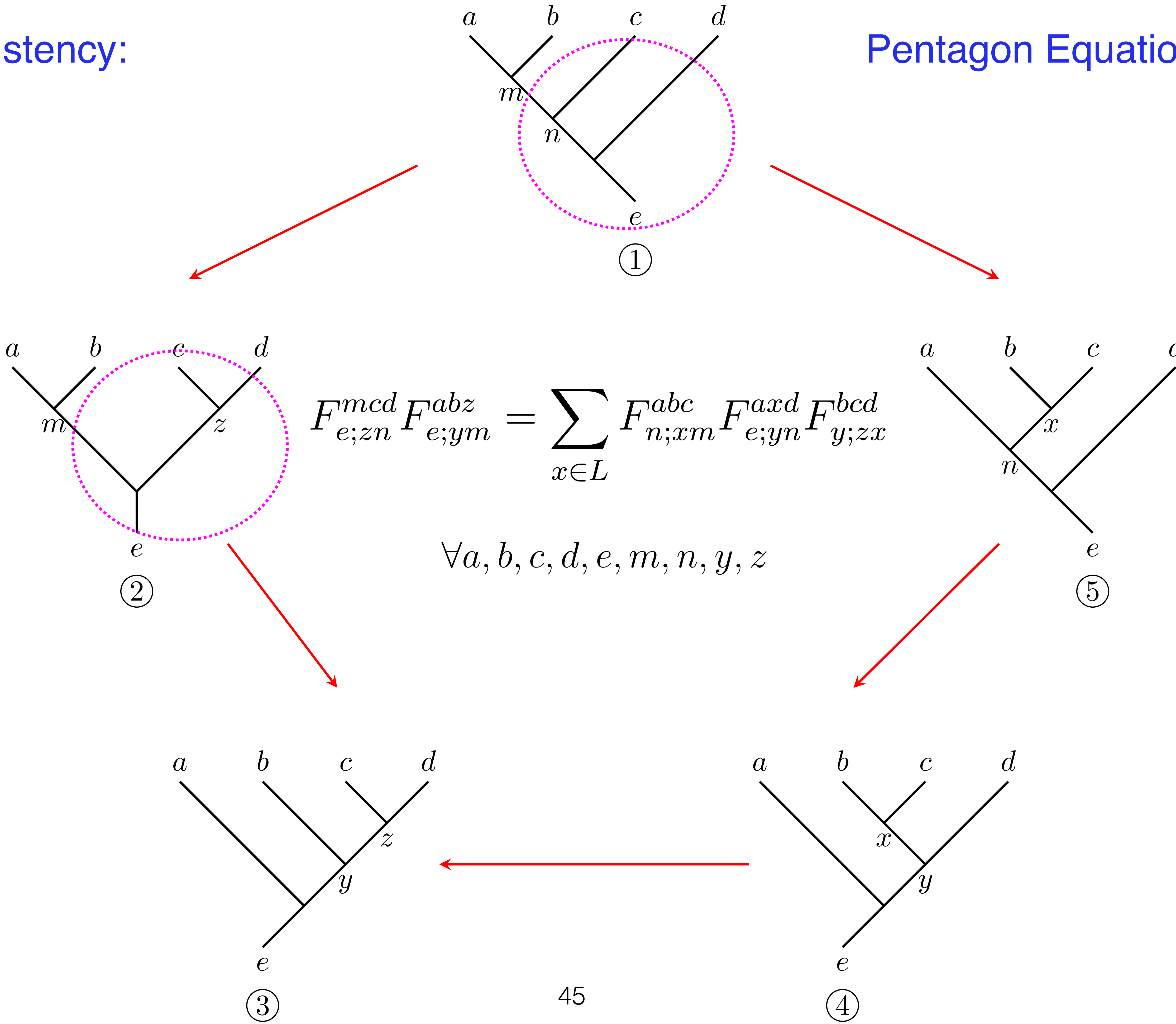


Consistency: **Pentagon Equation**



$$F^{mcd}_{e;zn} F^{abz}_{e;ym} = \sum_{x \in L} F^{abc}_{n;xm} F^{axd}_{e;yn} F^{bcd}_{y;zx}$$

$$\forall a, b, c, d, e, m, n, y, z$$

Pentagon equations guarantee consistency of basis change regardless of the particular sequence of F-moves.

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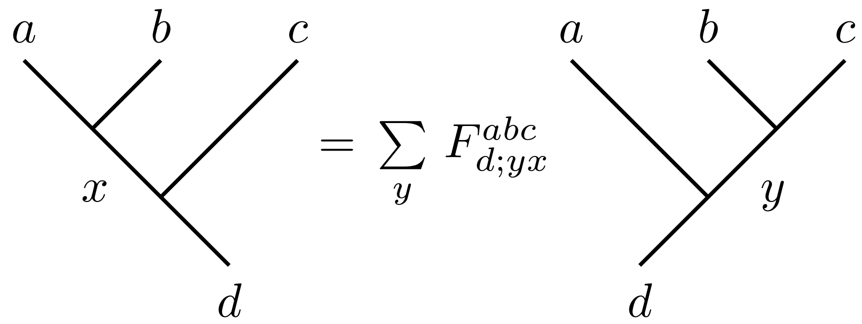

shorthand $\Longrightarrow$ notation

$$(\cdot\ \cdot)\cdot \xrightarrow{F} \cdot(\cdot\ \cdot)$$

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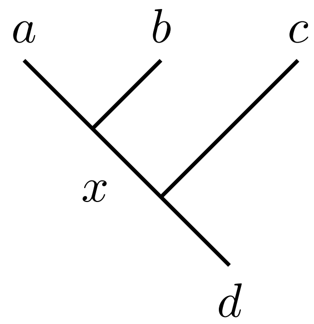

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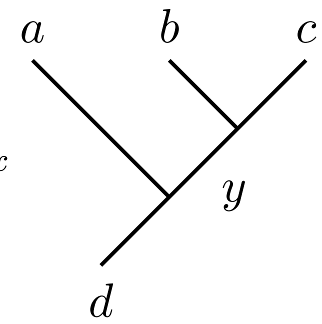

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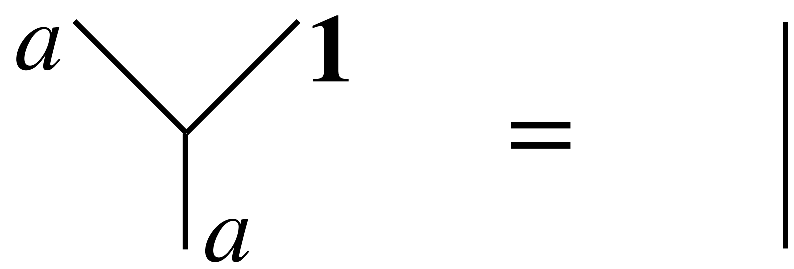
Exercise:

$$\begin{tikzcd}
 & & (((\cdot\cdot)\cdot)\cdot)\cdot \arrow[dl] \arrow[dr] & & \\
 & ((\cdot\cdot)\cdot)(\cdot\cdot) \arrow[dl] & & ((\cdot(\cdot\cdot))\cdot)\cdot \arrow[dr] & \\
(\cdot\cdot)(\cdot(\cdot\cdot)) \arrow[dr] & & & & (\cdot((\cdot\cdot)\cdot))\cdot \arrow[dl] \\
 & \cdot(\cdot(\cdot(\cdot\cdot))) & & (\cdot(\cdot(\cdot\cdot)))\cdot \arrow[d] & \\
 & & \cdot(\cdot((\cdot\cdot)\cdot)) \arrow[ul] & \cdot((\cdot(\cdot\cdot))\cdot) \arrow[l] &
\end{tikzcd}$$

The special role of $\mathbf{1}$:

$V_c^{a_1,\cdots,a_{i-1},a_i,\cdots,a_n}$ and $V_c^{a_1,\cdots,a_{i-1},\mathbf{1},a_i,\cdots,a_n}$ are naturally identified.



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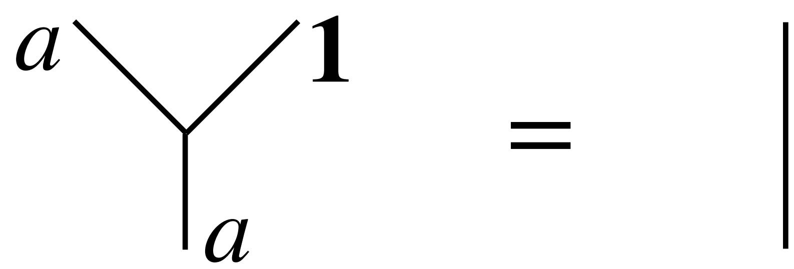
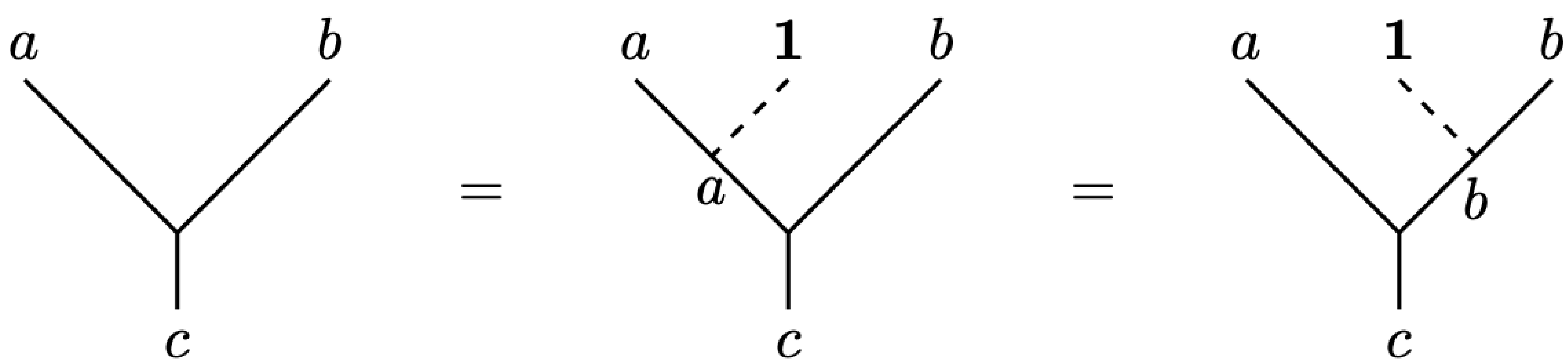


An edge labelled by $\mathbf{1}$ can be erased or inserted



$$\Longrightarrow \quad F^{a\mathbf{1}b}_{c;ba} = 1$$

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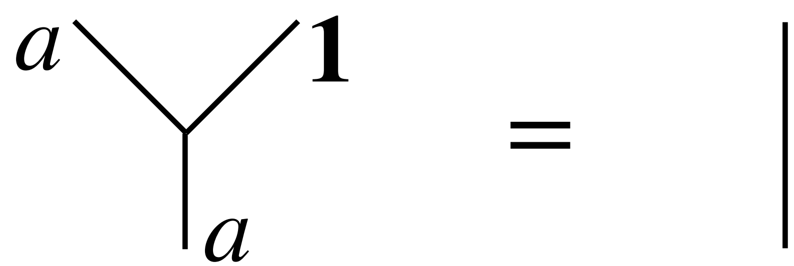

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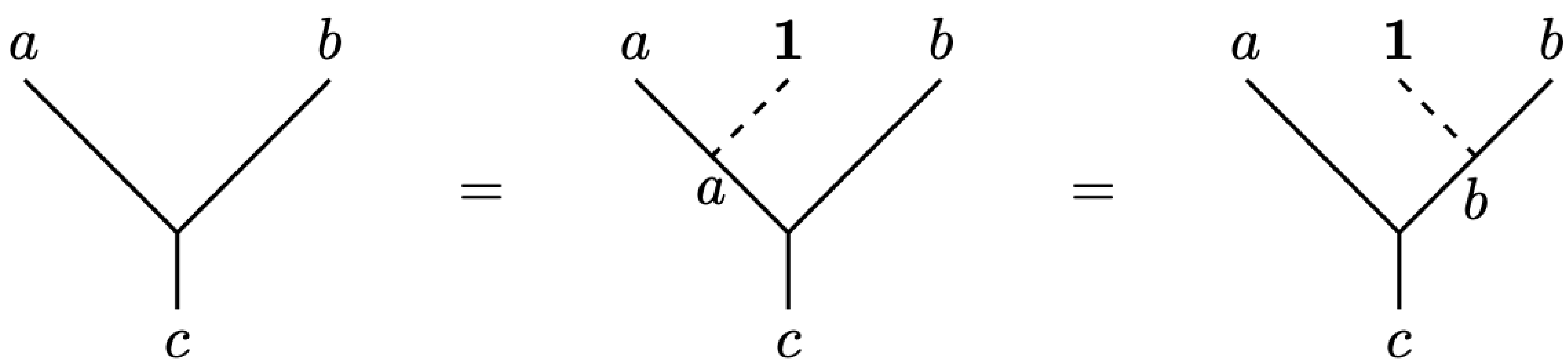


$$\Longrightarrow \quad F^{a\mathbf{1}b}_{c;ba} = 1$$

In fact, it follows that $F^{abc}_d = (1)$ whenever one of a, b, c is $\mathbf{1}$

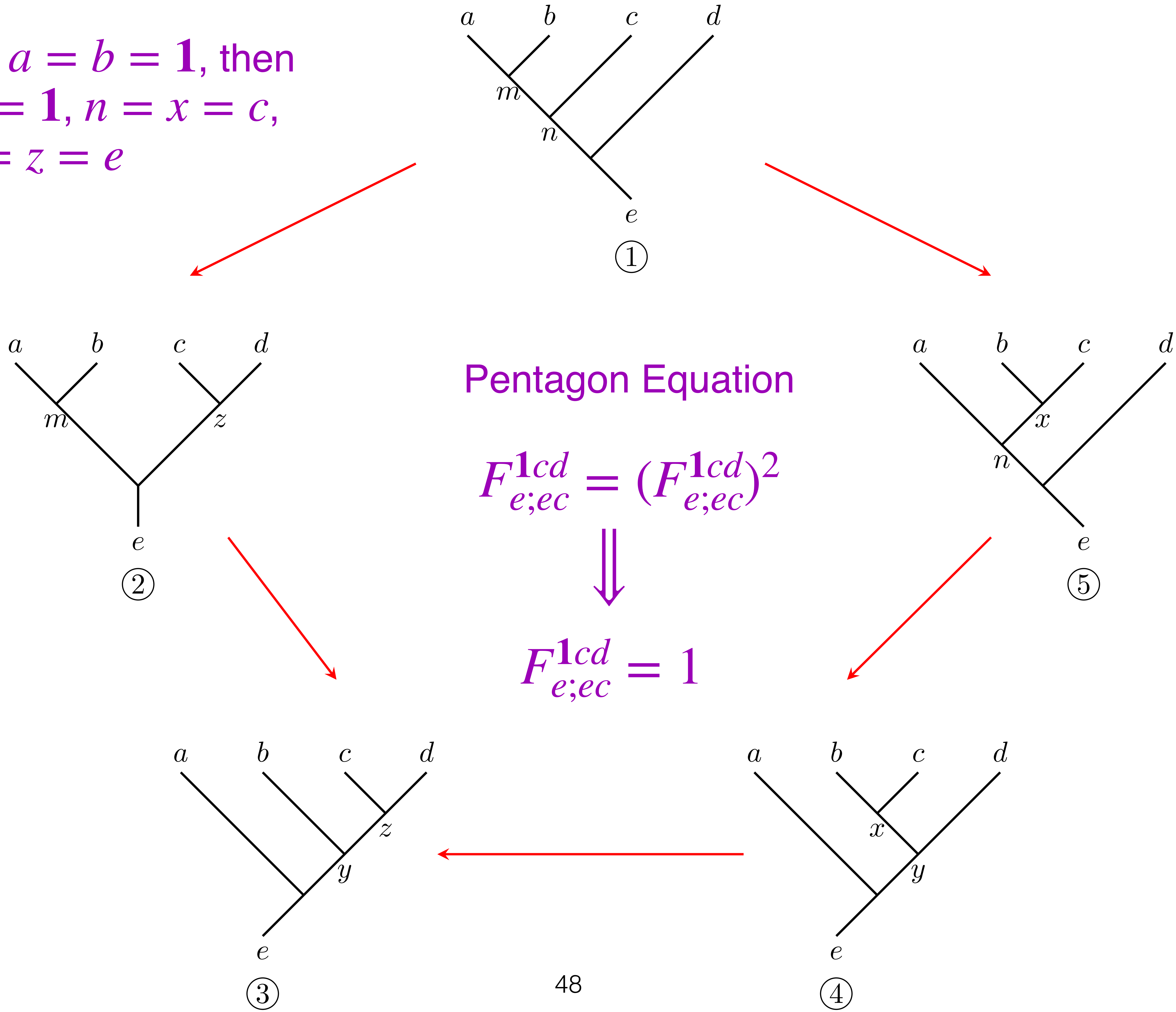
set $a = b = \mathbf{1}$, then
$m = \mathbf{1}$, $n = x = c$,
$y = z = e$

Pentagon Equation

$$F^{\mathbf{1}cd}_{e;ec} = (F^{\mathbf{1}cd}_{e;ec})^2$$

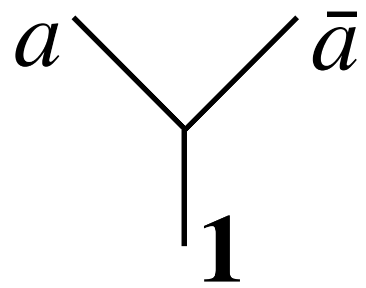
$$\Downarrow$$

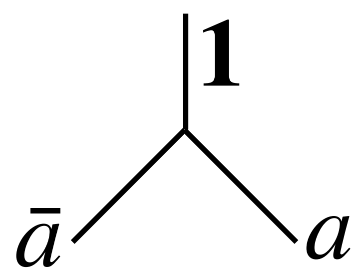
$$F^{\mathbf{1}cd}_{e;ec} = 1$$



★ Creation/annihilation

Creating/fusing a pair of dual anyons from/to the vacuum:





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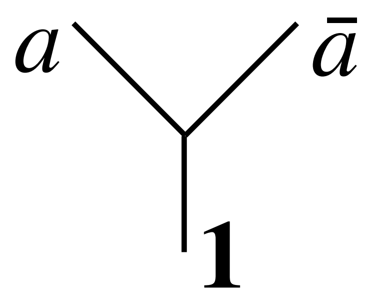
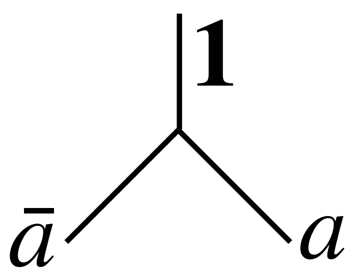
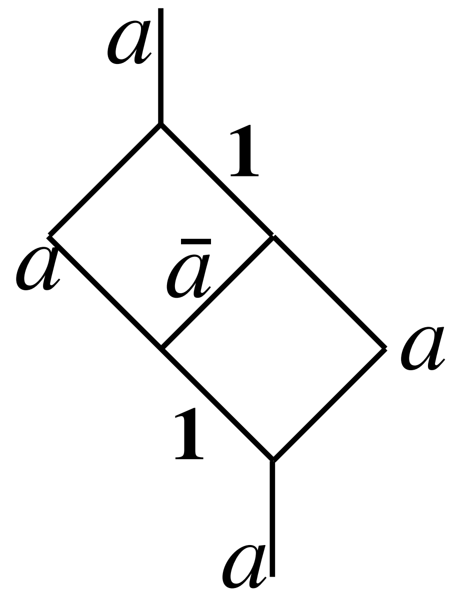




Consider:



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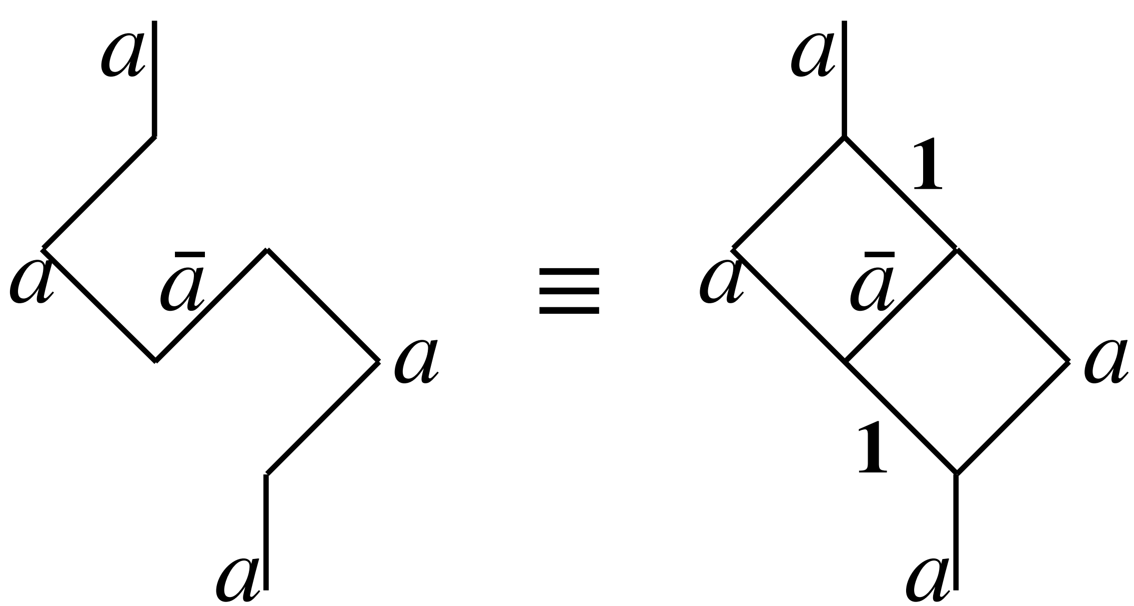

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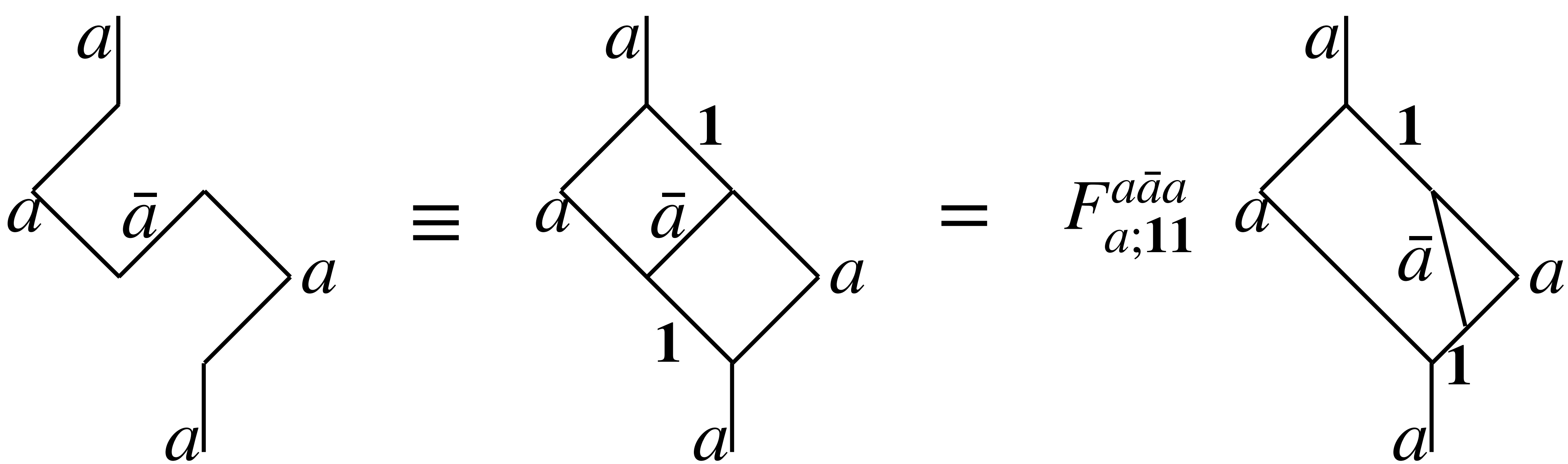

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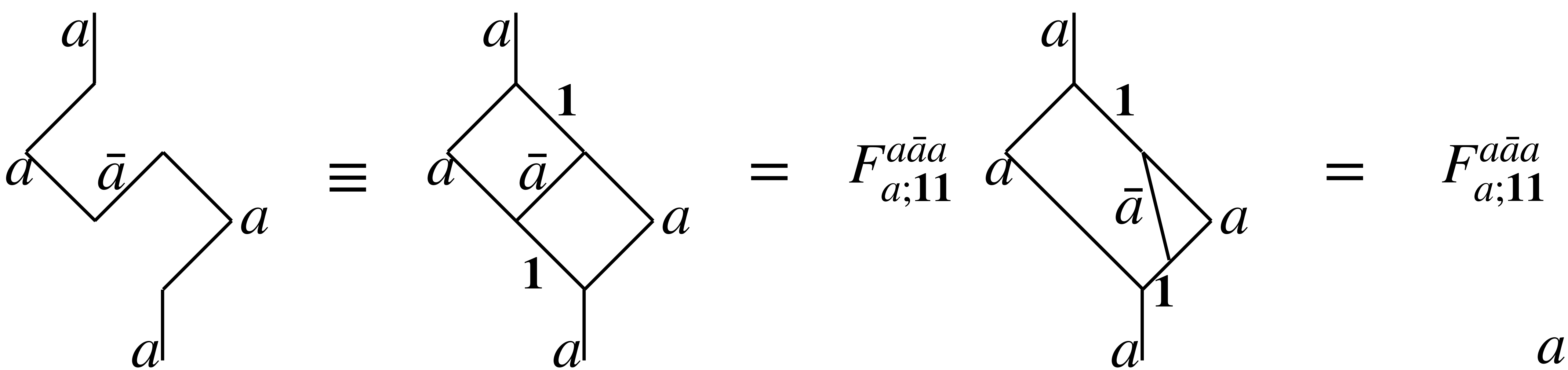
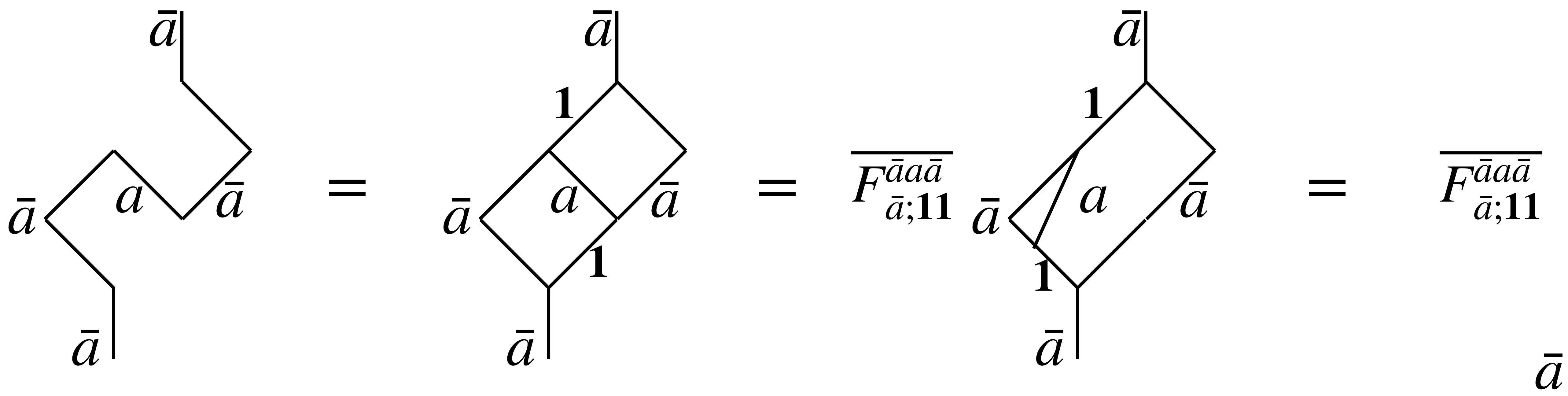


Similarly,



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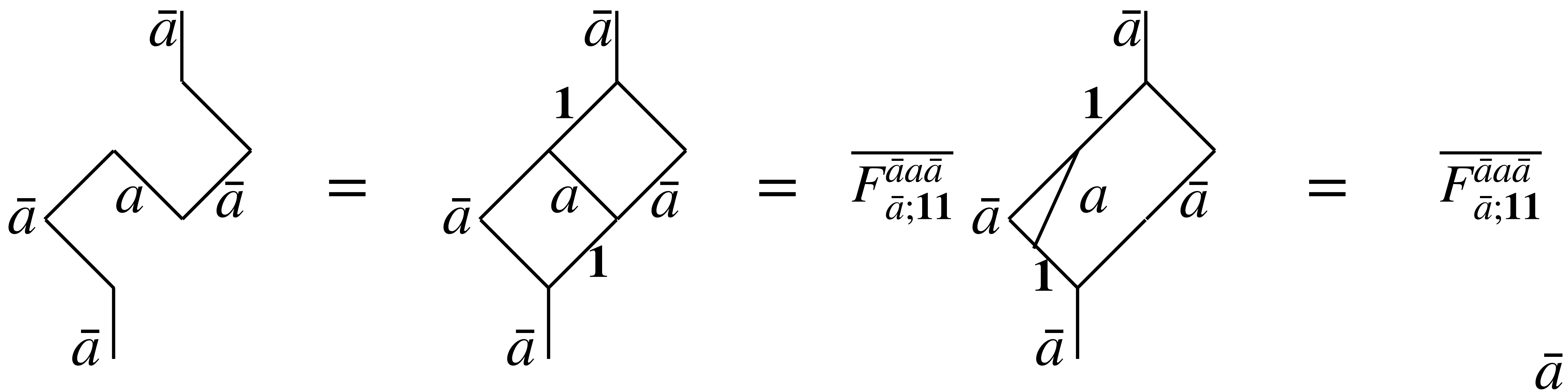
Similarly,



Renormalize creation/fusion pair to have homotopy invariance

$$\Longrightarrow \quad F^{a\bar{a}a}_{a;\mathbf{11}} = \overline{F^{\bar{a}a\bar{a}}_{\bar{a};\mathbf{11}}} \neq 0$$

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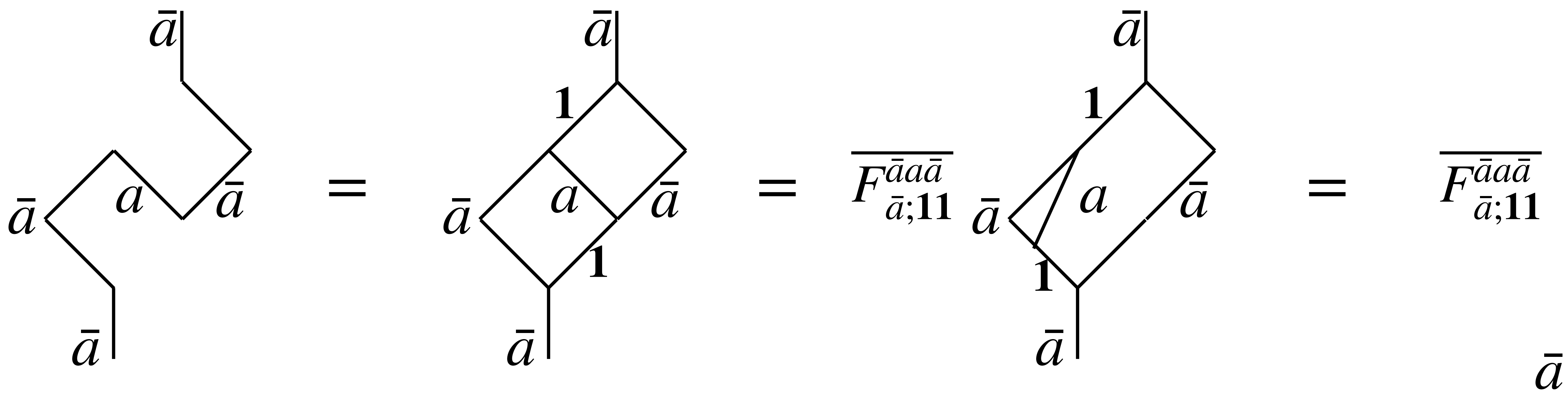

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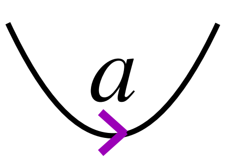
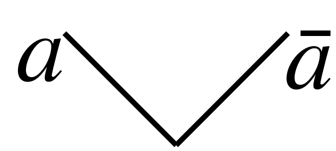


$:= \phi_a$



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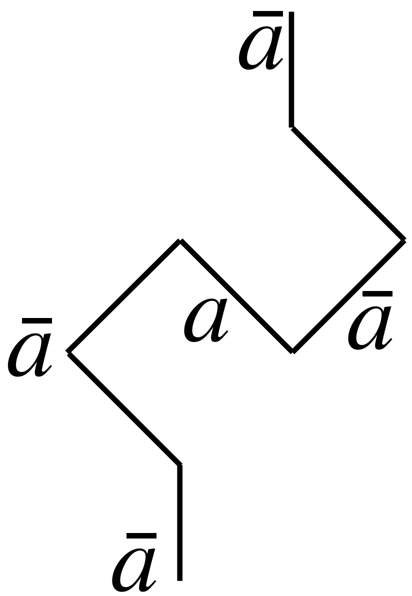

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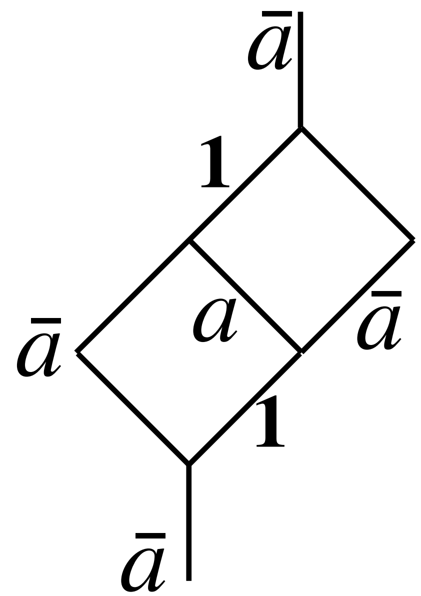
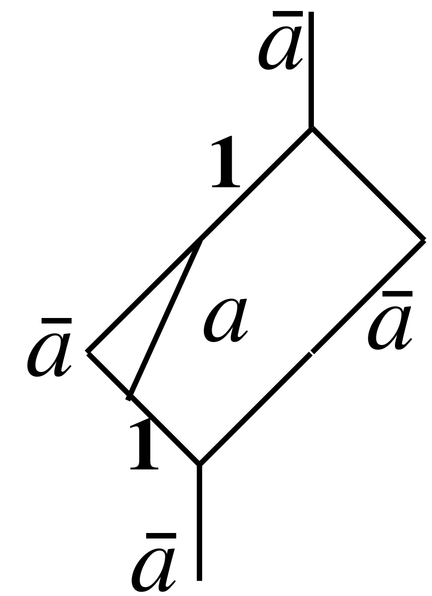


$= \overline{F^{\bar{a}a\bar{a}}_{\bar{a};\mathbf{11}}}$



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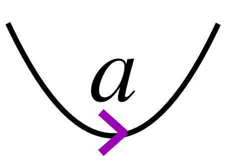

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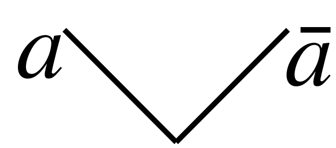

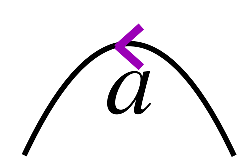

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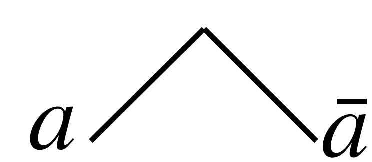

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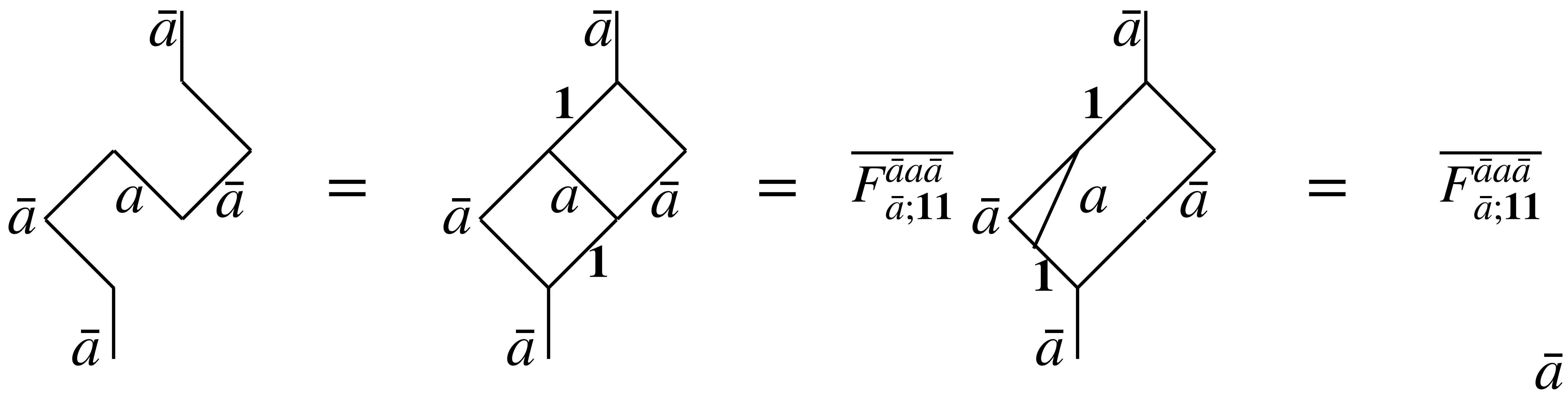

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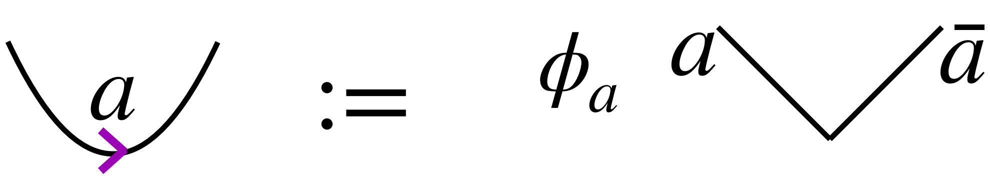

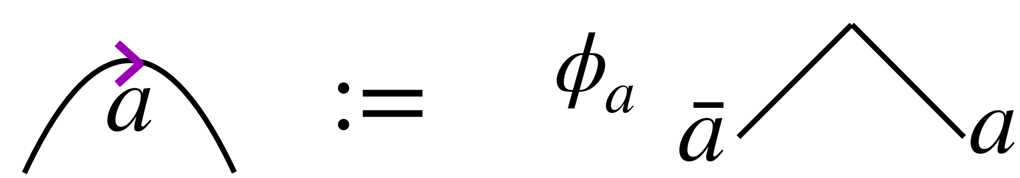

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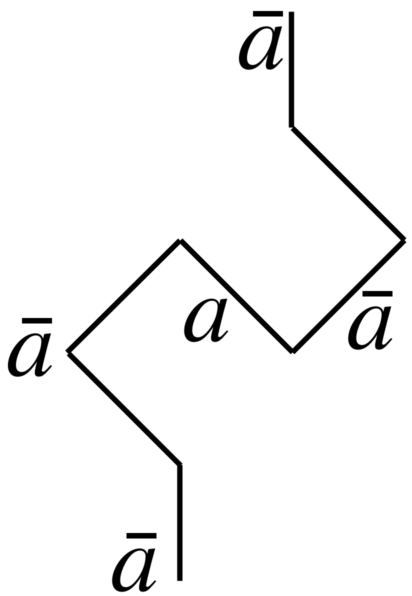

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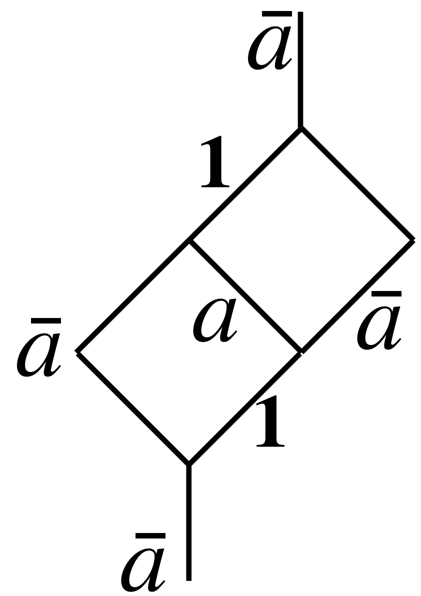

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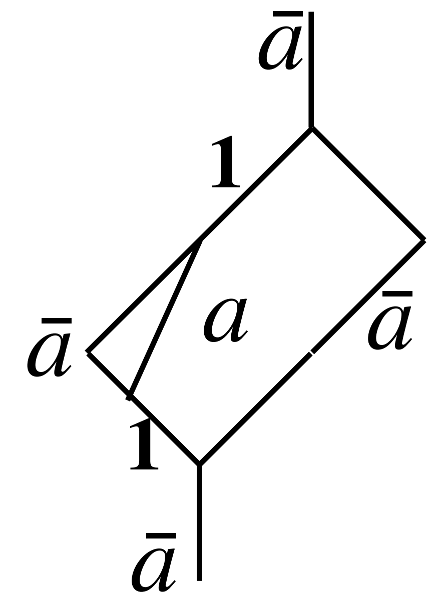

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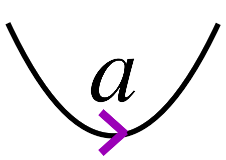
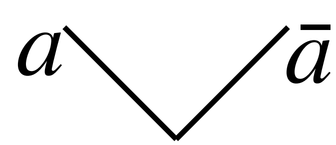
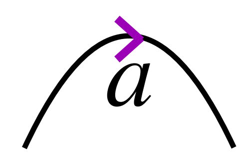


$:= \phi_a$





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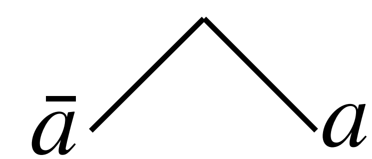
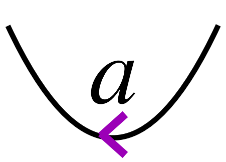
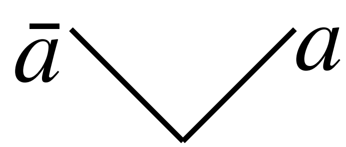
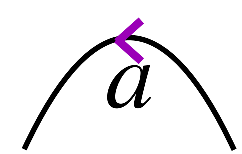




$:= \overline{\phi_a}$





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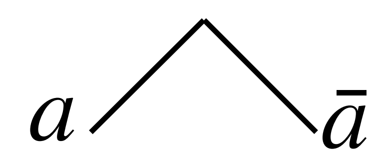

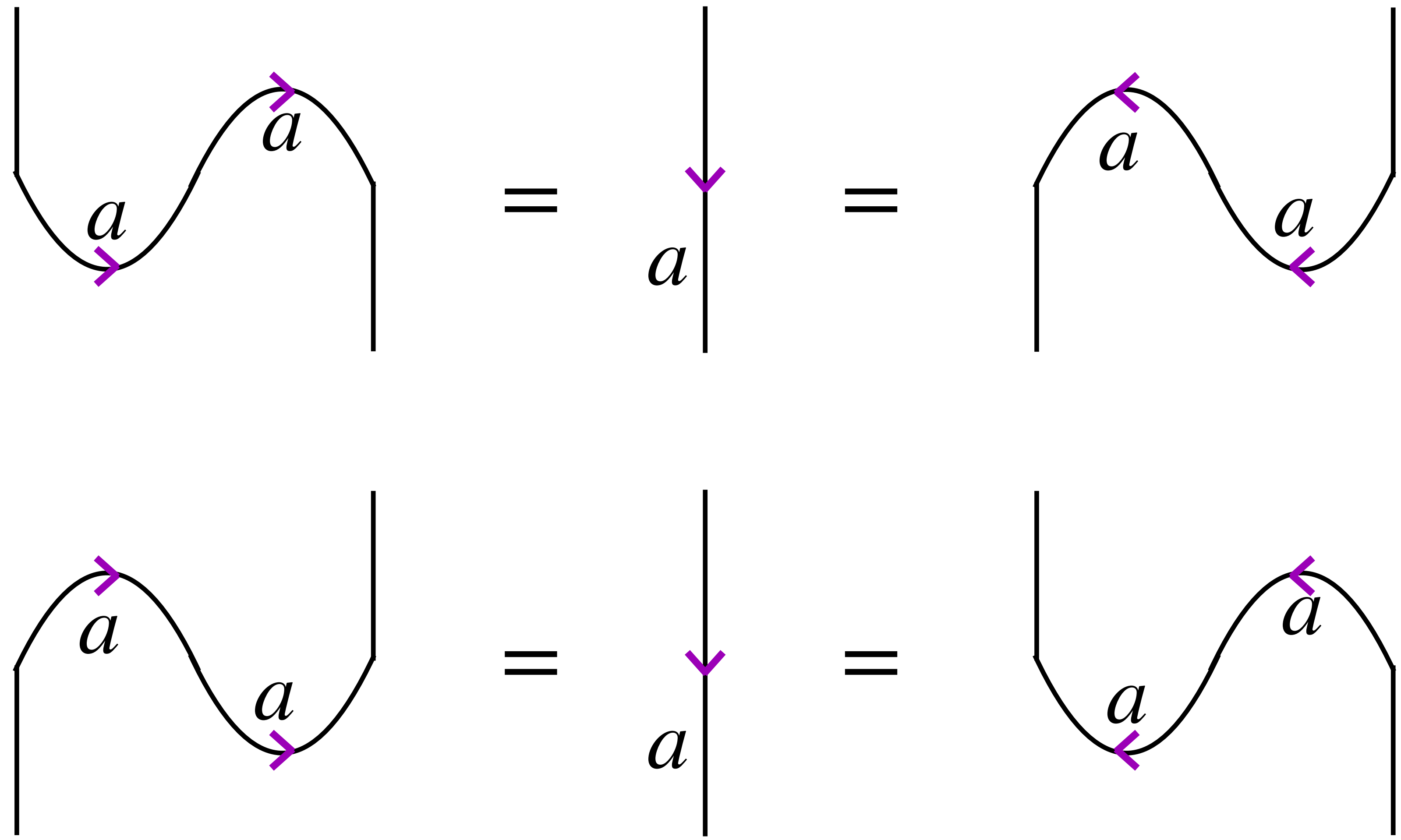
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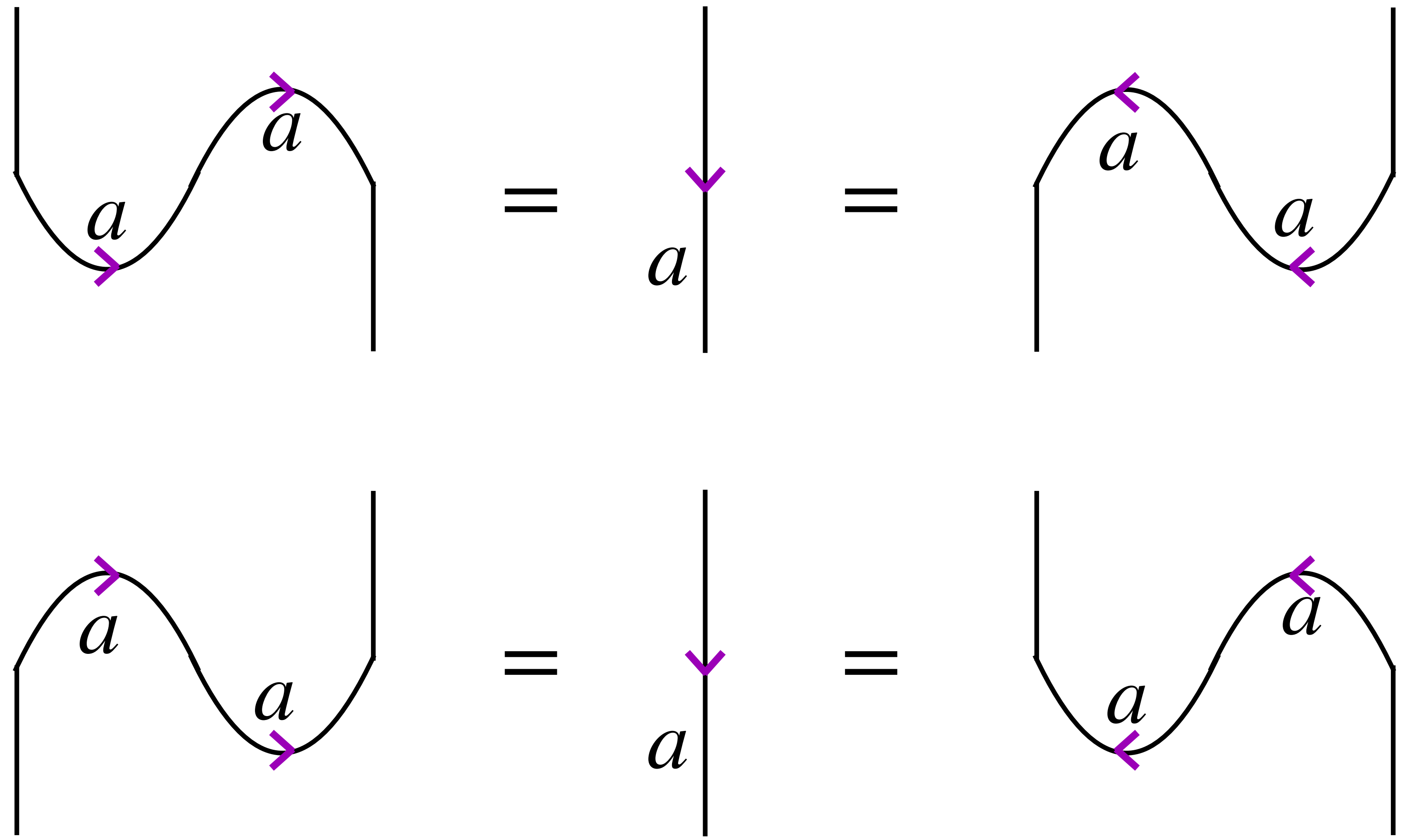
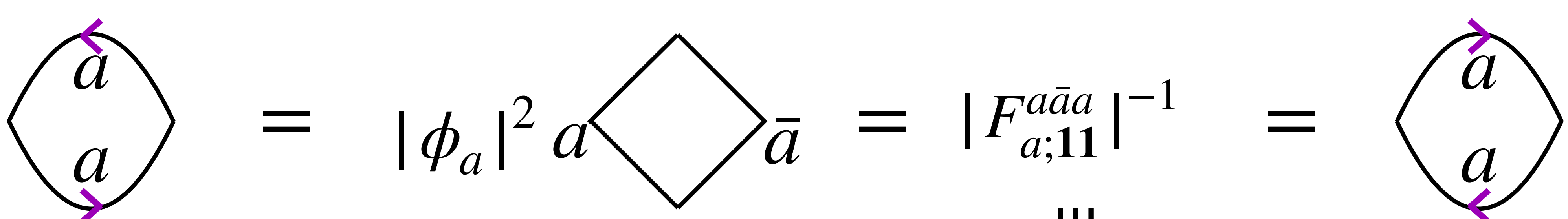


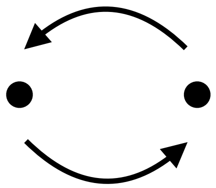
Moreover,

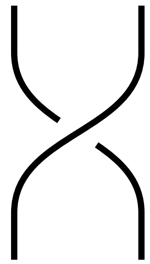


$|F^{a\bar{a}a}_{a;\mathbf{11}}|^{-1} = \dim_q a$

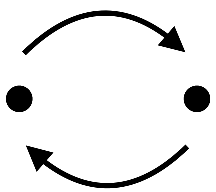
quantum dimension $\dim_q a$

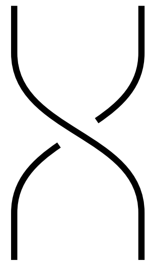
★ Braiding





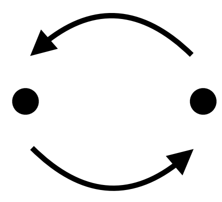
counterclockwise

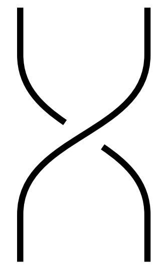




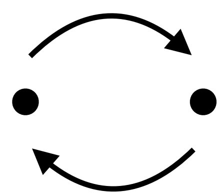
clockwise   spacetime process

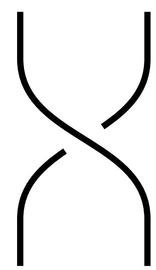
★ Braiding





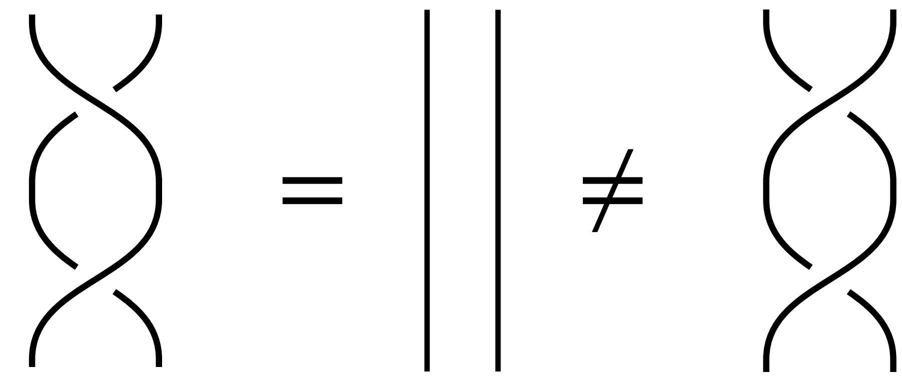
counterclockwise





clockwise

spacetime process



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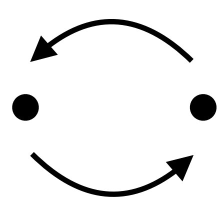
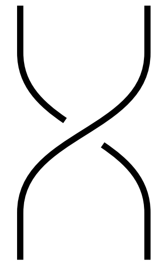
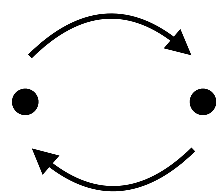
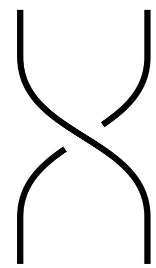


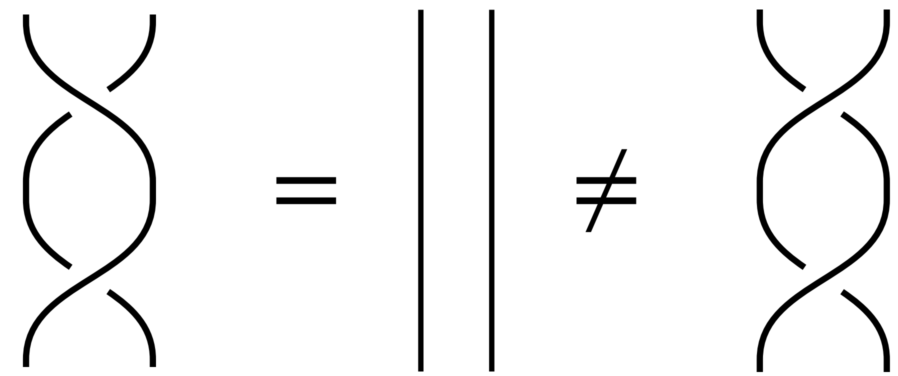


counterclockwise





clockwise   spacetime process



Braiding induces unitary maps: $V_c^{a_1,\cdots,a_{i-1},a_i,\cdots,a_n} \longrightarrow V_c^{a_1,\cdots,a_i,a_{i-1},\cdots,a_n}$

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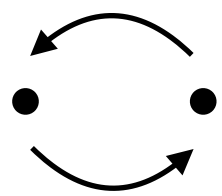
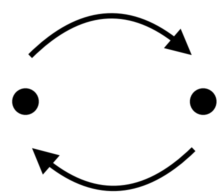


counterclockwise



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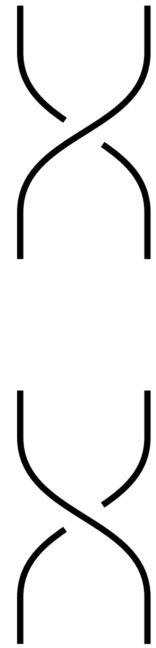
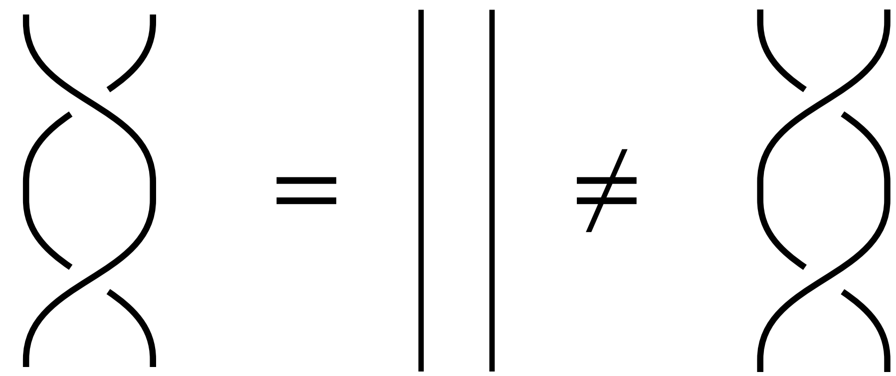


spacetime process



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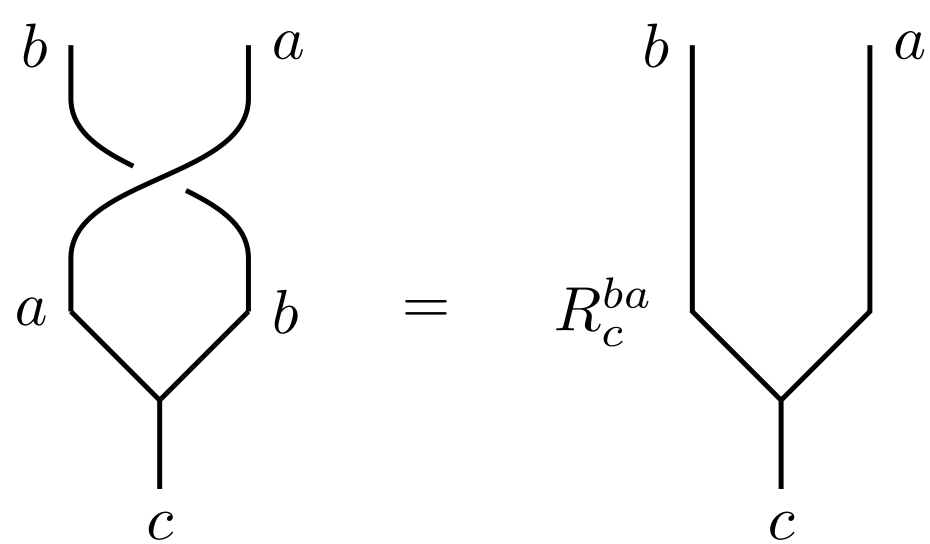
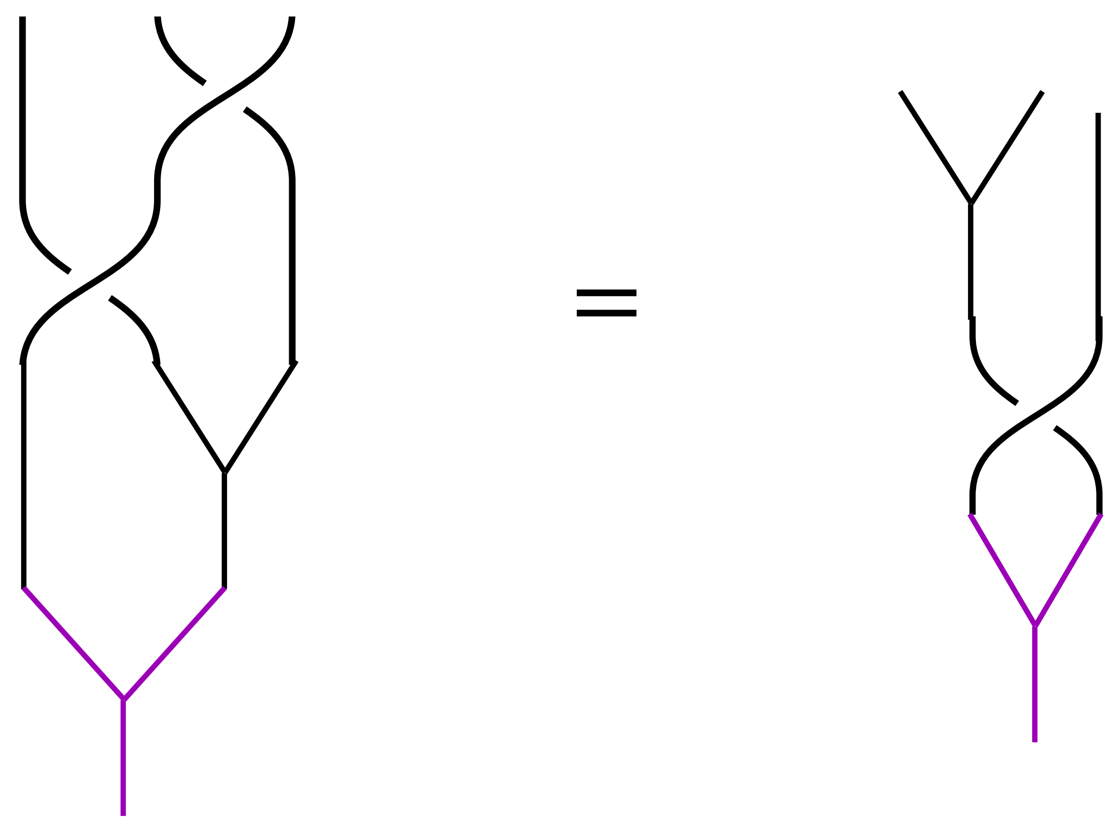


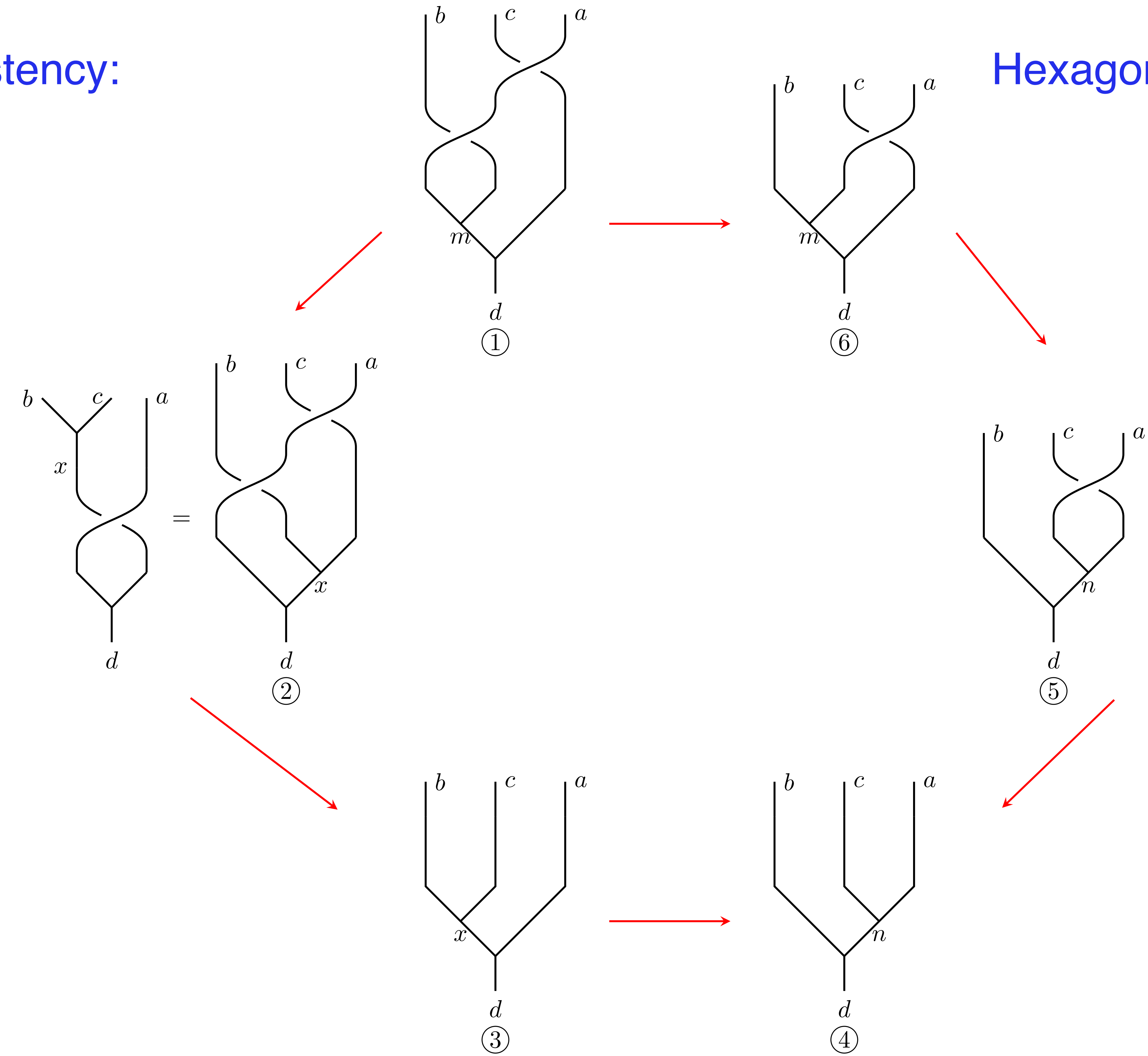
R-move

R-symbols are phases since they are unitary

Naturality: braiding commutes with physical processes



Consistency: Hexagon Equation 1



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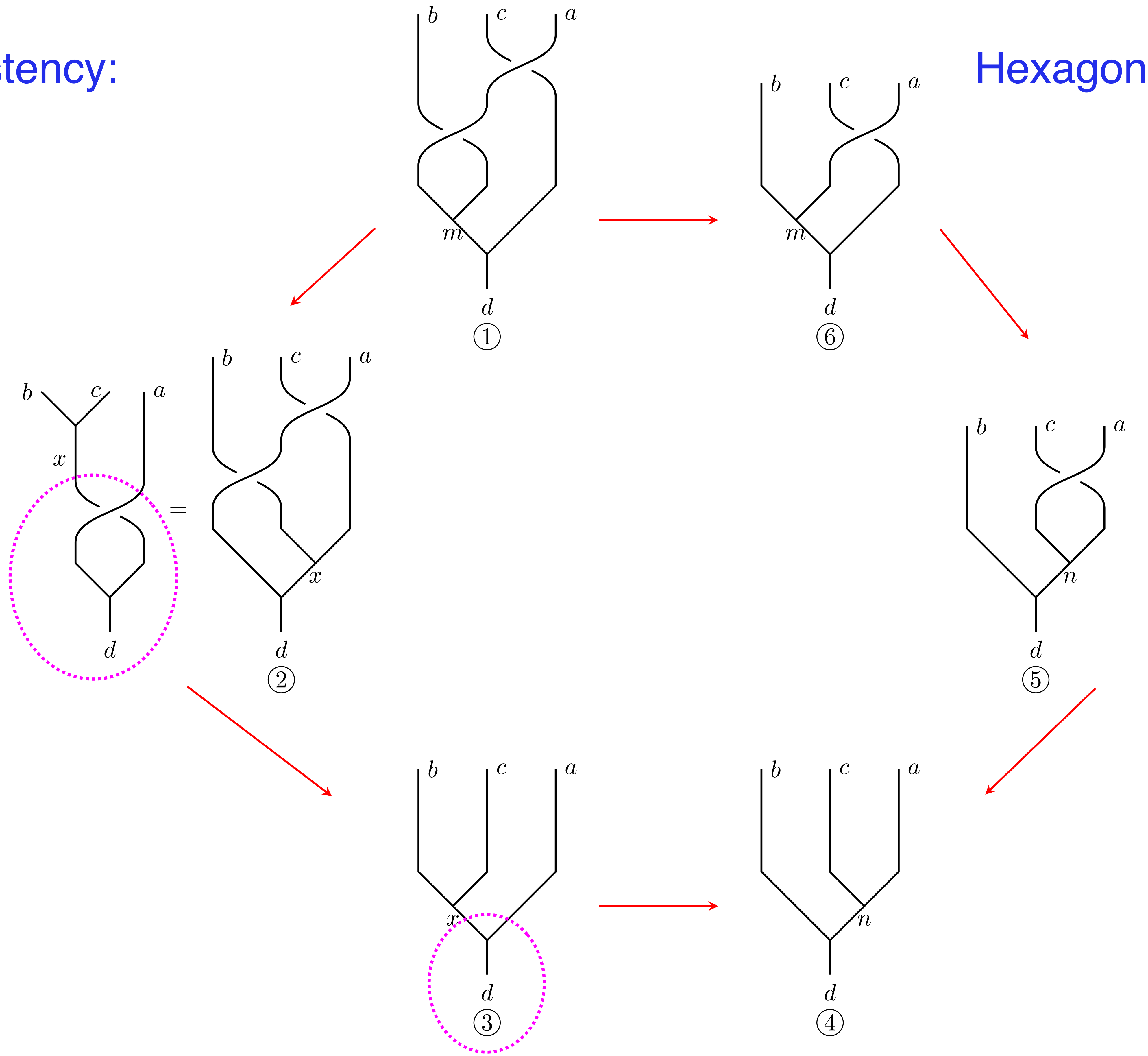

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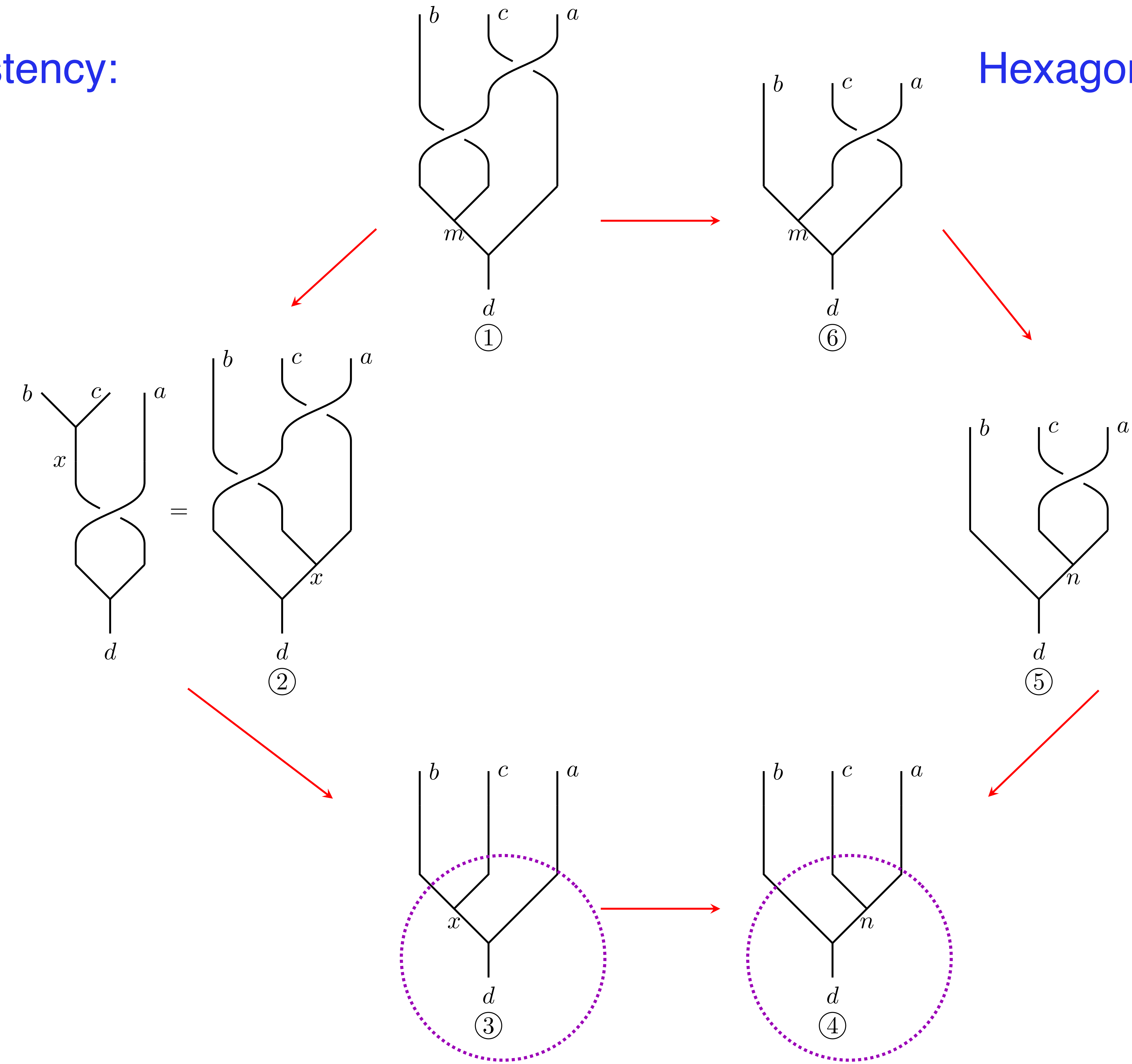

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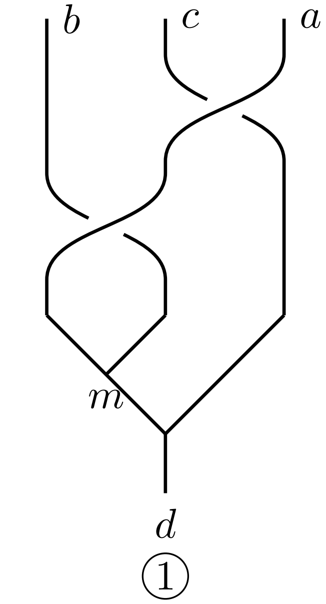
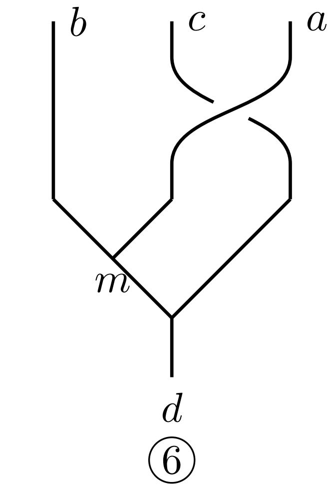
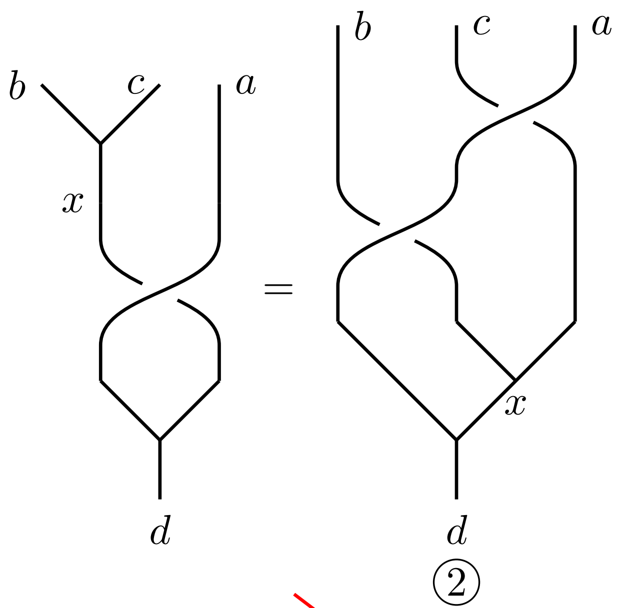
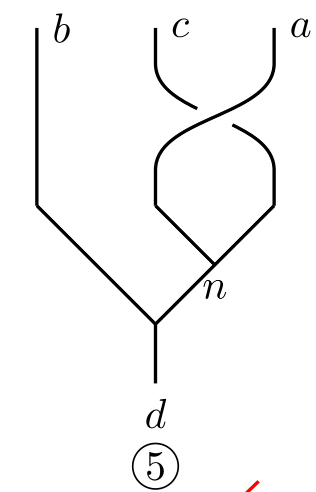


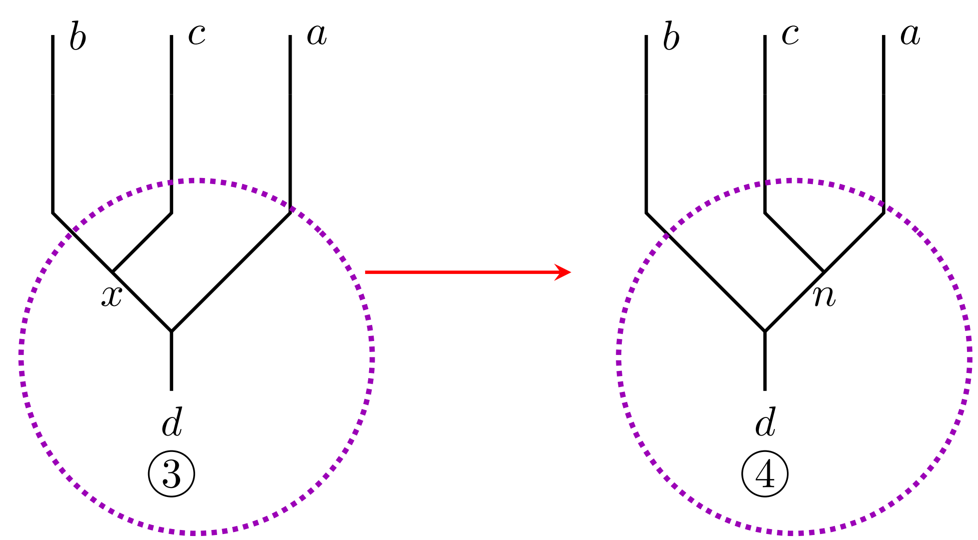




$$R^{ba}_{m} F^{bac}_{d;nm} R^{ca}_{n} = \sum_{x \in L} F^{abc}_{d;xm} R^{xa}_{d} F^{bca}_{d;nx}$$

$$\forall a, b, c, d, m, n$$





Similarly, replacing braiding by the inverse braiding:

$$(R^{ab}_m)^{-1} F^{bac}_{d;nm} (R^{ac}_n)^{-1} = \sum_{x \in L} F^{abc}_{d;xm} (R^{ax}_d)^{-1} F^{bca}_{d;nx} .$$

Hexagon Equation 2

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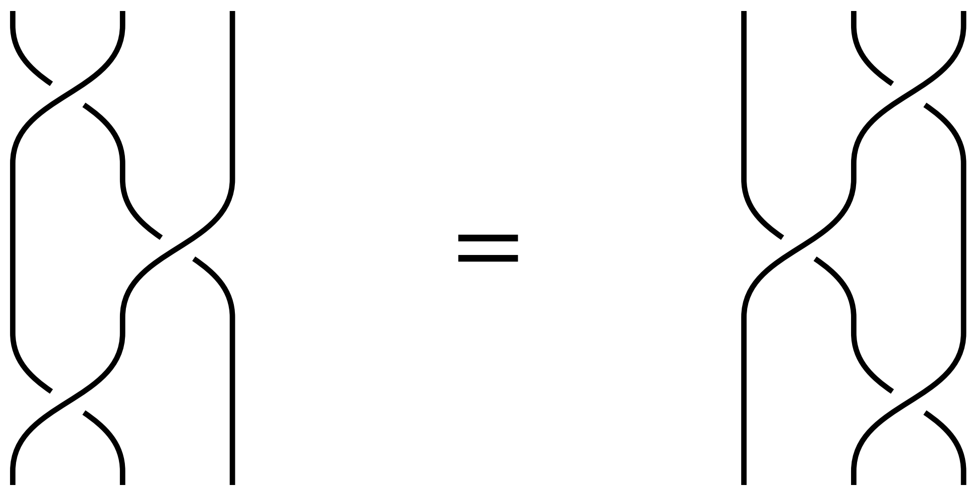
The Yang-Baxter Equation follows:



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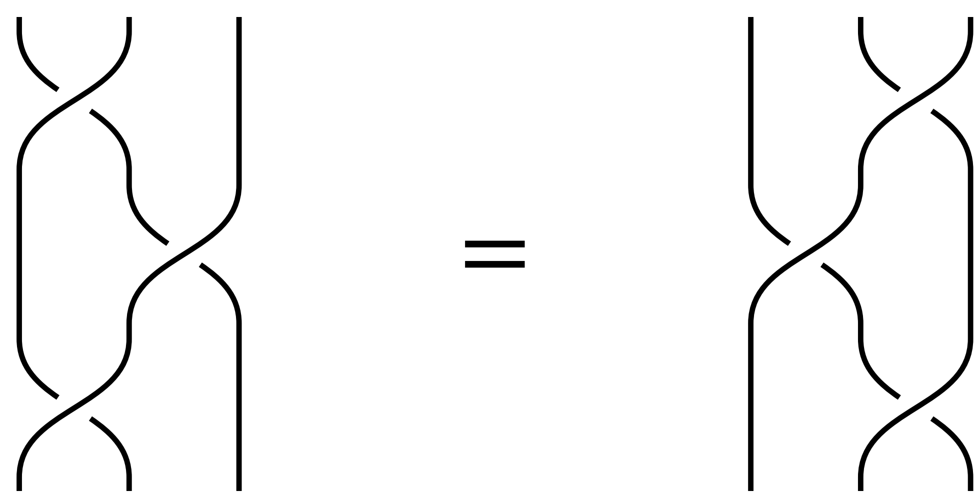

It also follows: $R^{a\mathbf{1}}_{a} \; = \; R^{\mathbf{1}a}_{a} \; = \; 1$

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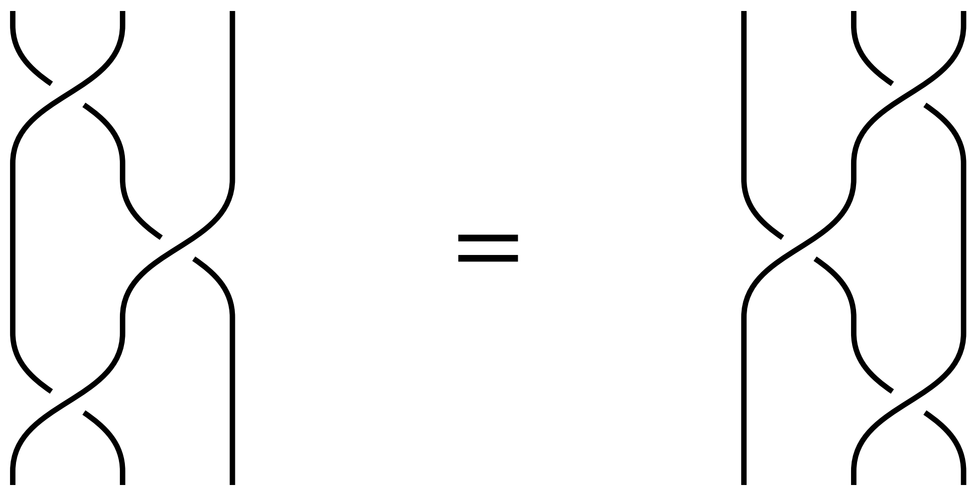

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Example: Fib $\qquad R^{\tau\tau}_1 = e^{-4\pi i/5}, \quad R^{\tau\tau}_\tau = e^{3\pi i/5}$

★ Summary

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$$\{L,\ N_{ab}^{c},\ F_{d,ef}^{abc}\} \quad \text{satisfying}$$

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- Fusion rule constraints
- *F*-symbols constrains:
  - Pentagon equations with unitary *F*-matrices
  - Compatibility with unit: $F^{a\mathbf{1}b}_{c;ba} = 1$
  - Compatibility with duals: $F^{a\bar{a}a}_{a;\mathbf{11}} = \overline{F^{\bar{a}a\bar{a}}_{\bar{a};\mathbf{11}}} \neq 0$

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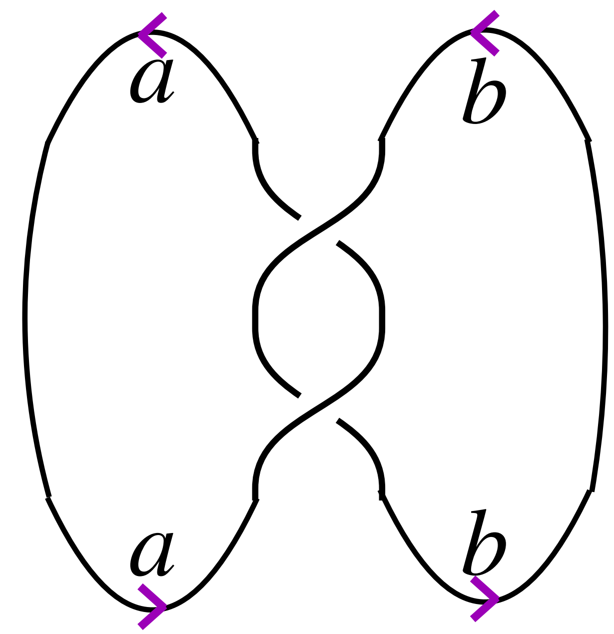
A (multiplicity-free) unitary modular tensor category (UMTC) is a UBFC with a non-degenerate S-matrix:

$$S = \left(S_{ab}\right)_{a,b\in L} \qquad\qquad S_{ab} \quad :=$$



A (multiplicity-free) anyon system is characterized by a UMTC

★ More examples

Ising

- Label set: $L = \{\mathbf{1}, \psi, \sigma\}$; $\bar{a} = a$.
- Fusion rules: $\psi \otimes \psi = \mathbf{1}$, $\psi \otimes \sigma = \sigma$, $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$.
- F-matrices:

$$F_{\sigma}^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F_{\sigma}^{\psi\sigma\psi} = F_{\psi}^{\sigma\psi\sigma} = (-1).$$

- R-symbols: $R_{\mathbf{1}}^{\psi\psi} = -1, \quad R_{\sigma}^{\psi\sigma} = R_{\sigma}^{\sigma\psi} = -i, \quad R_{\mathbf{1}}^{\sigma\sigma} = e^{-\pi i/8}, \quad R_{\psi}^{\sigma\sigma} = e^{3\pi i/8}.$

Toric code

- Label set: $L = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\} \longleftrightarrow \{\mathbf{1}, e, m, f\}$; $\bar{a} = a$.
- Fusion rules: $a \otimes b = (a+b) \mod 2$.
- F-matrices: (1).
- R-symbols: $R_{a+b}^{a,b} = (-1)^{a_1 b_2}$, $a = (a_1, a_2), b = (b_1, b_2) \in L$.

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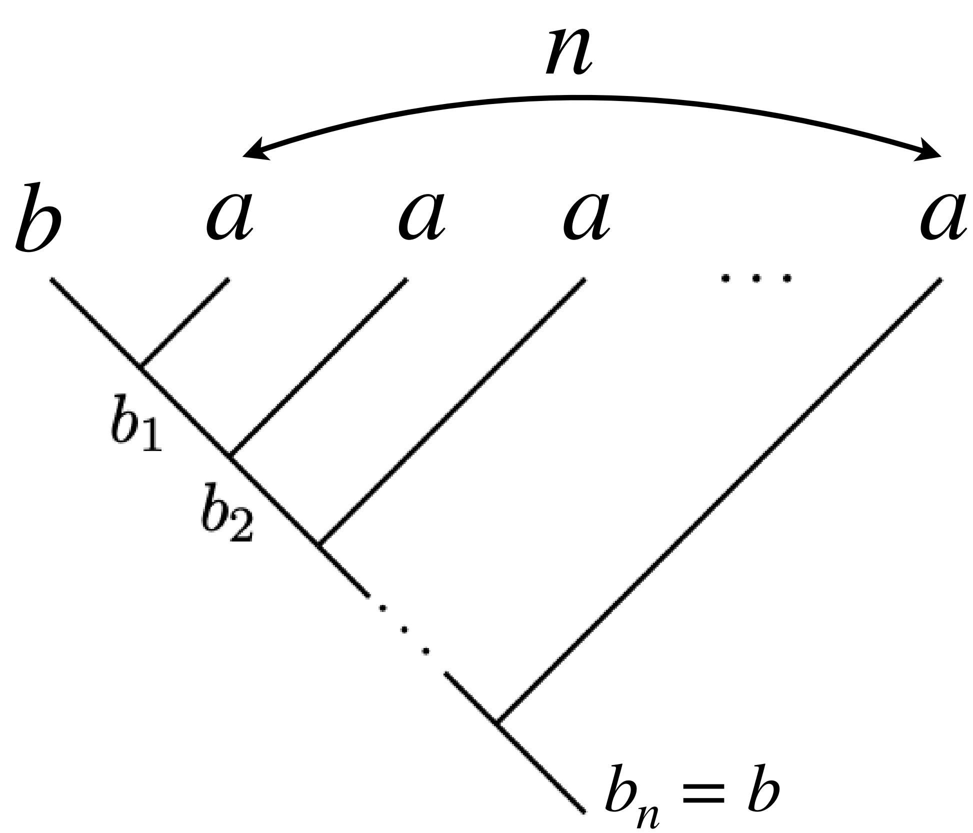
- $\dim_q a$ is the Fronenius-Perron dimension of a, that is, the largest eigenvalue, dominating in absolute value, of the matrix N_a, with $(N_a)_{bc} := N^c_{ab}$

Consider the space $V_b^{b,a,\cdots,a}$



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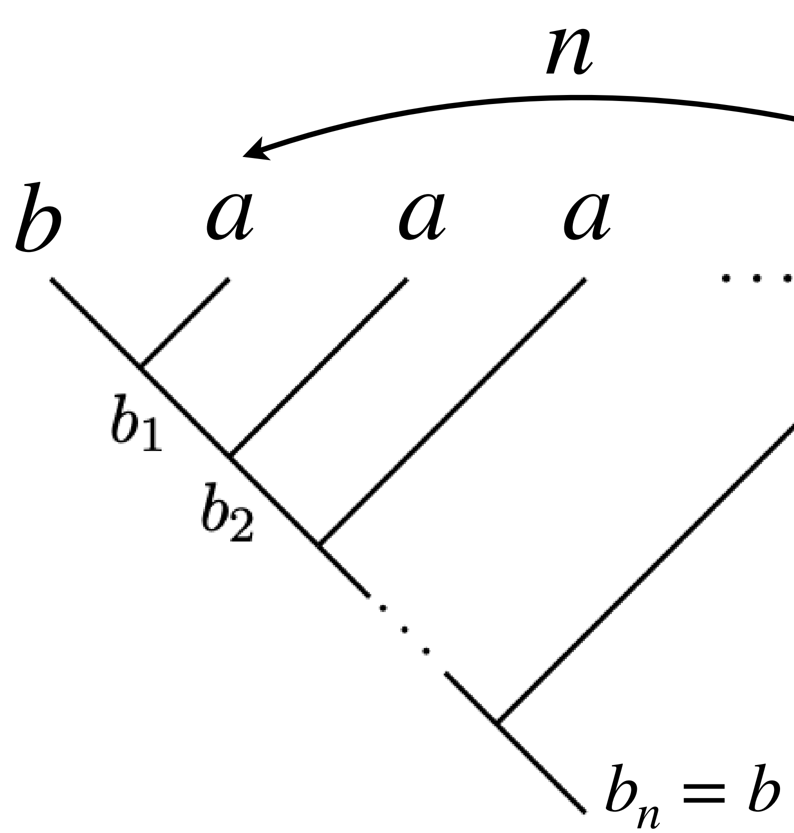

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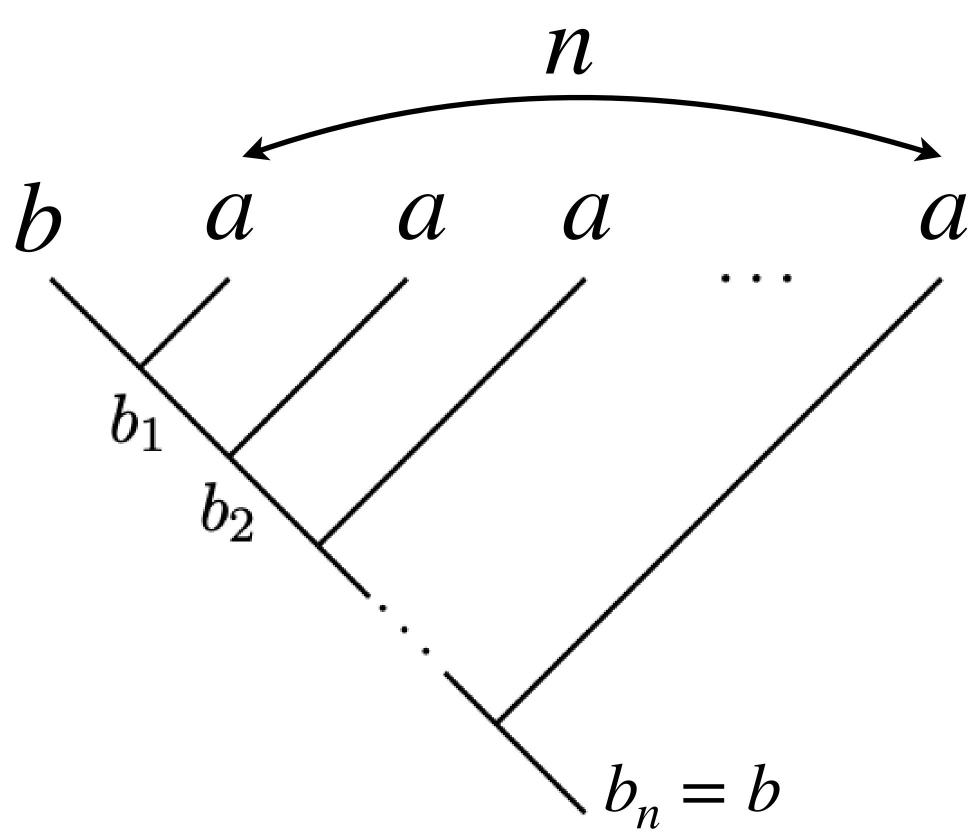

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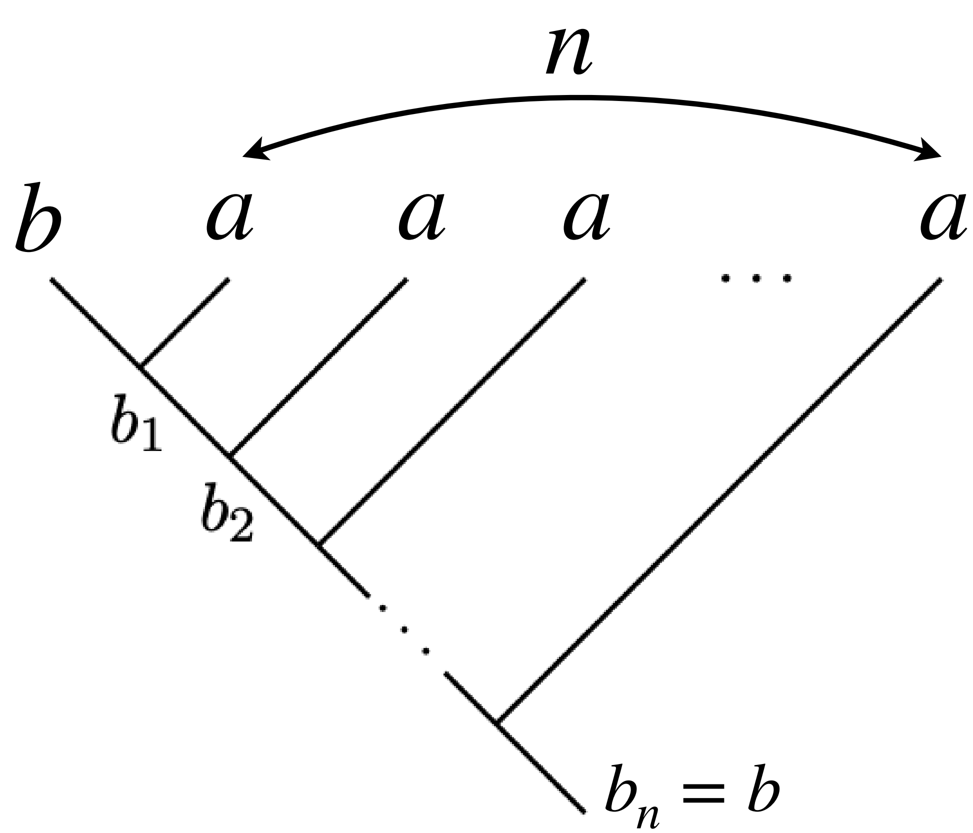

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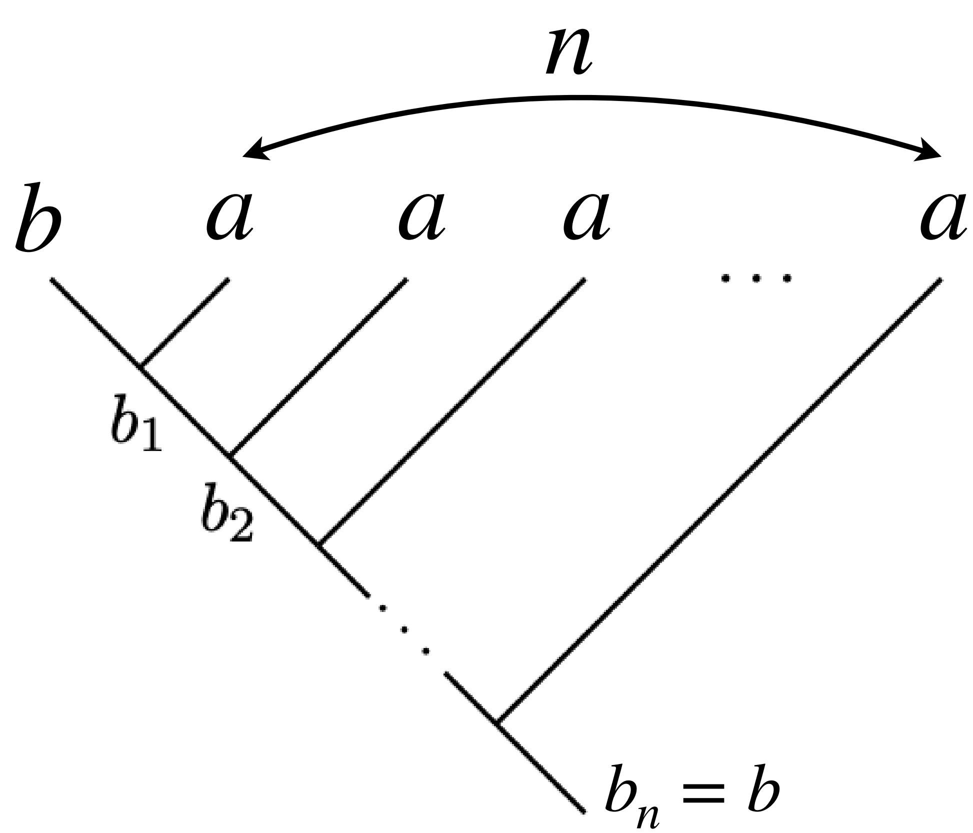

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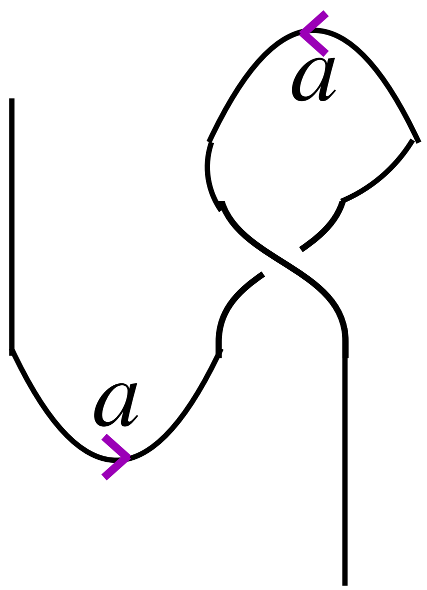
Exercise: Fib

Fibonacci sequence $\{F_n\}$: $F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0,\ F_1 = 1.$

$$\dim V_\tau^{\tau^{\otimes n}} = \dim V_{\mathbf{1}}^{\tau^{\otimes(n+1)}} = F_n \qquad \dim_q \tau = \frac{1+\sqrt{5}}{2}$$

★ Some related quantities: topological twist



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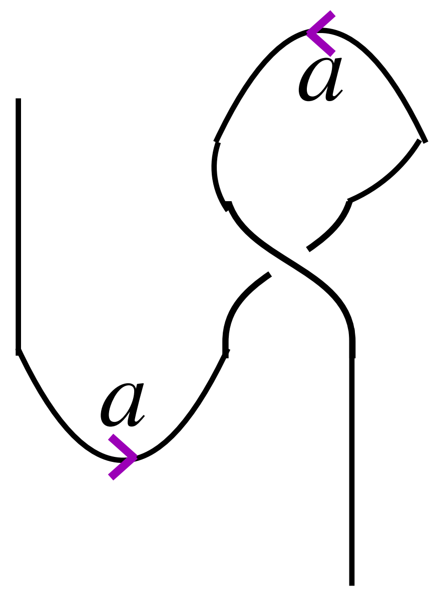

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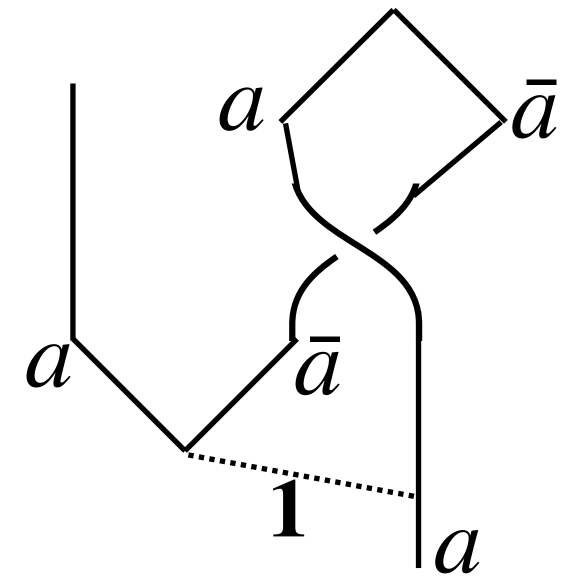

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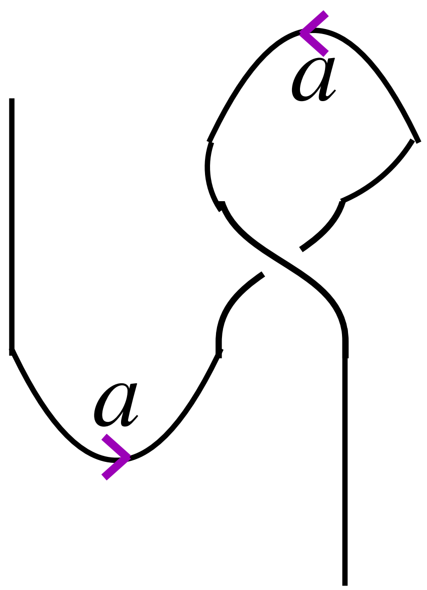

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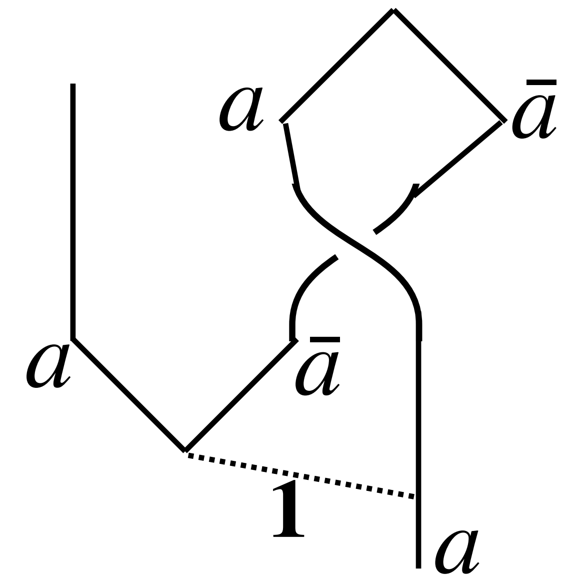

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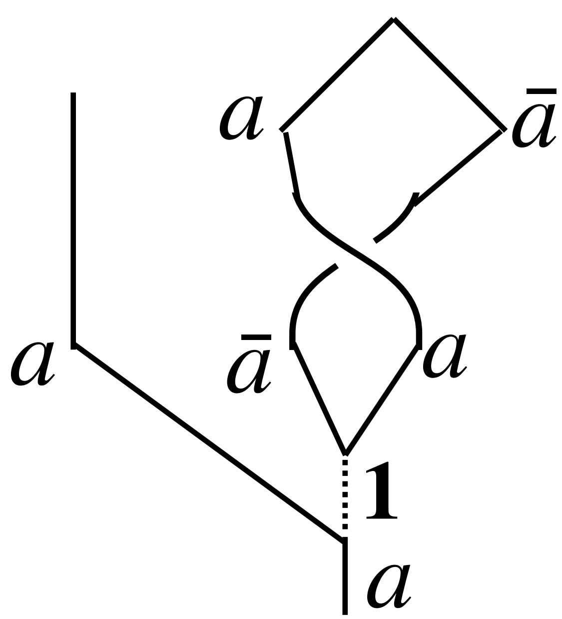

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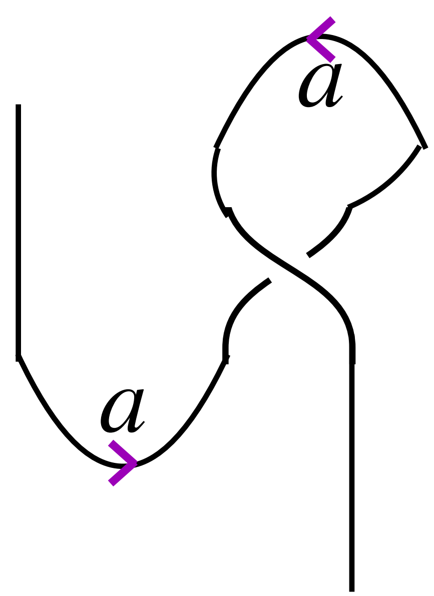
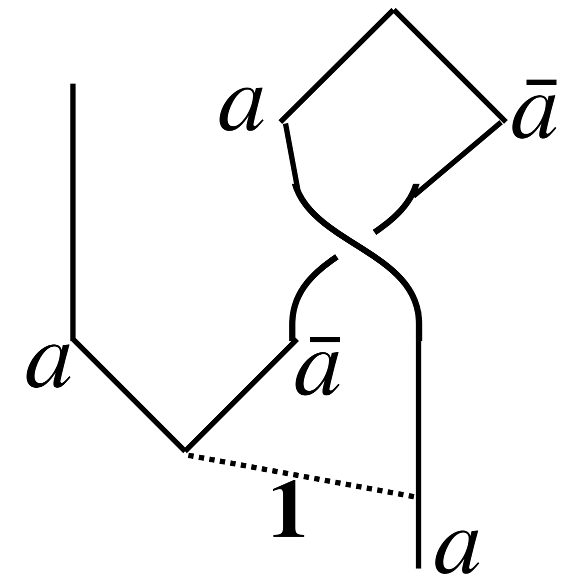


$=$ $\dim_q(a)$ 

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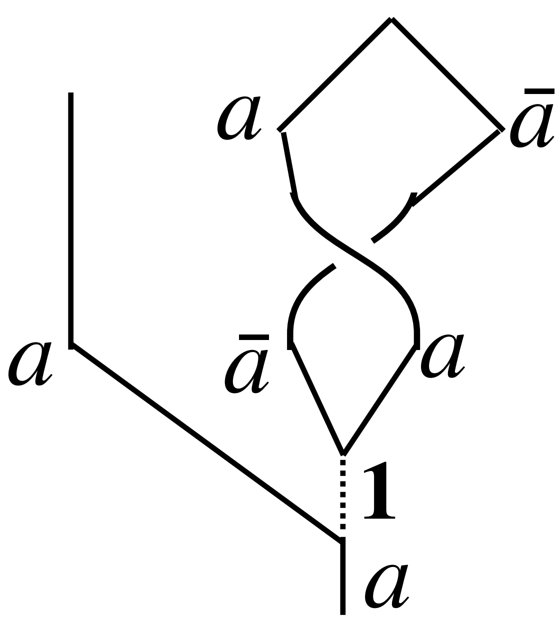
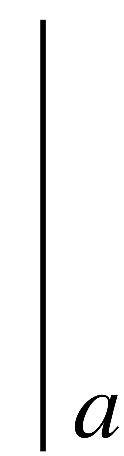
$\dim_q(a)\, F^{a\bar{a}a}_{a;\mathbf{11}}$ 

$= \quad F^{a\bar{a}a}_{a;\mathbf{11}}\ \dim_q(a)\ \overline{R^{\bar{a}a}_{\mathbf{1}}}$ 

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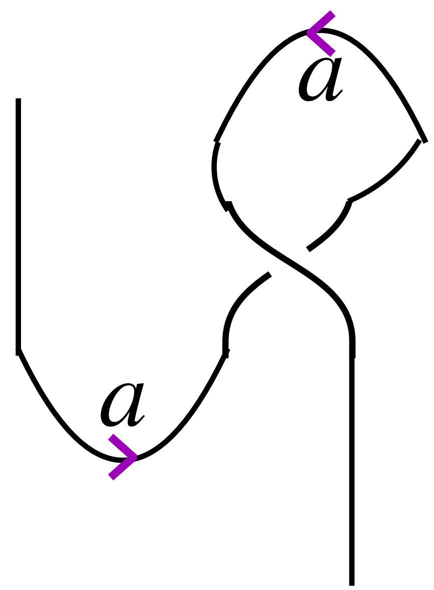

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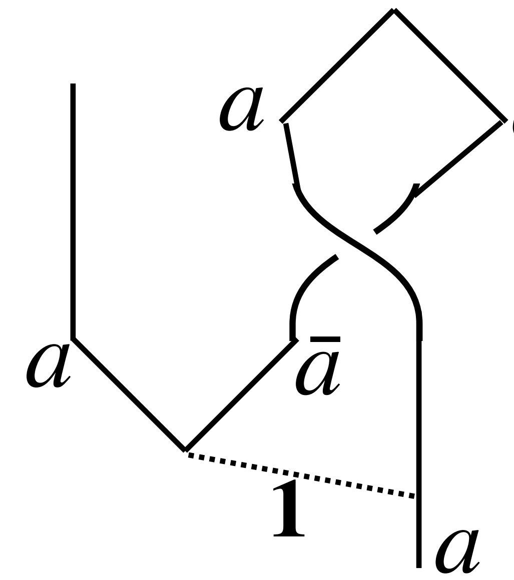

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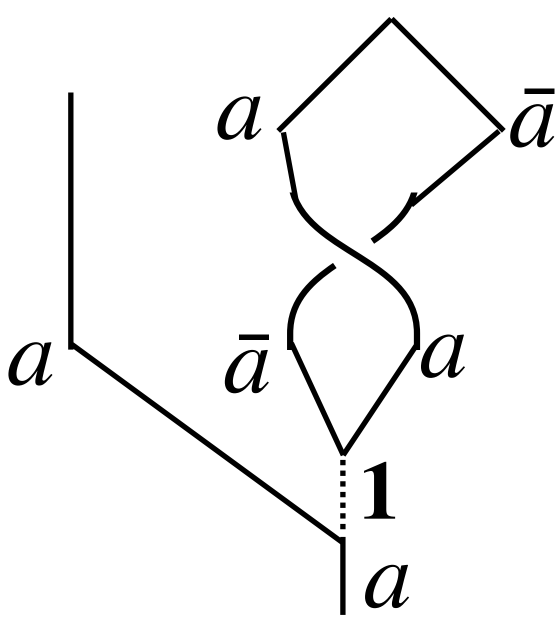

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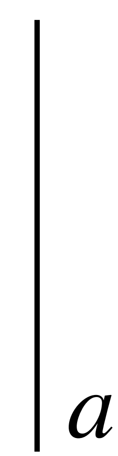

|||

topological twist θ_a

★ Some related quantities: braid group representation

N-strand braid group

$$B_N = \langle \sigma_1, \cdots, \sigma_N \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \ \sigma_i \sigma_j = \sigma_j \sigma_i, |\, i-j \,| \geq 2 \ \rangle$$

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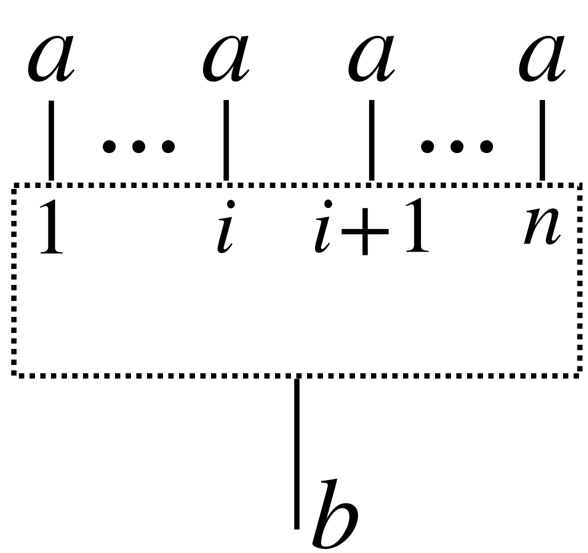
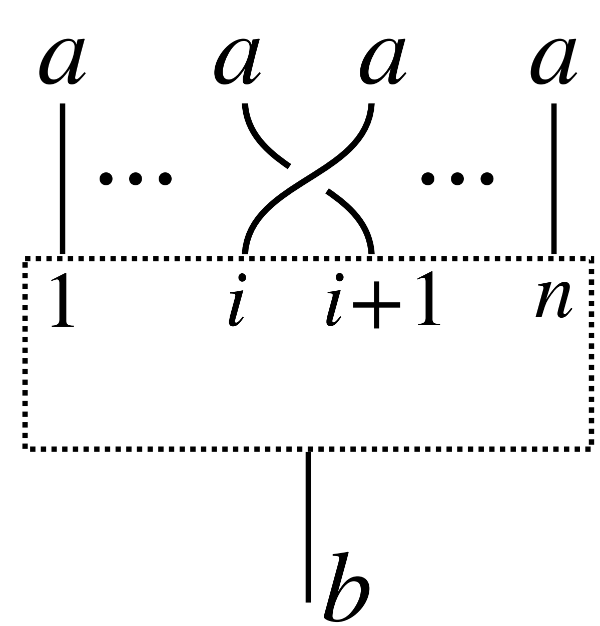
$\rho_b^{a,n}(\sigma_i)$:



$\longmapsto$



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- is endowed with a non-degenerate braiding

$$V \otimes W \simeq W \otimes V$$

UMTC in abstract category language:

anyon language	category language
anyon type	simple object
multiple anyons	tensor product $\otimes$
superposition of anyons	direct sum $\oplus$
state space $V_b^{a_1,\cdots,a_n}$	Hom space $\mathrm{Hom}(b, a_1 \otimes \cdots \otimes a_n)$
physical processes	morphisms
anyon braiding	isomorphism: $a \otimes b \cong b \otimes a$
fusing rules	direct sum decomposition