

An introduction to Topological Quantum Computing

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Mini course for
Algebraic structures in topology 2026

University of Puerto Rico, Río Piedras, July 10-17, 2026

①

(physics)
topological phase of matter



topological
quantum computing

②

(algebra)
higher category theory



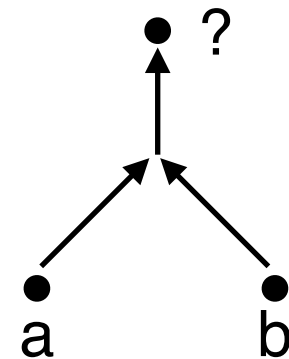
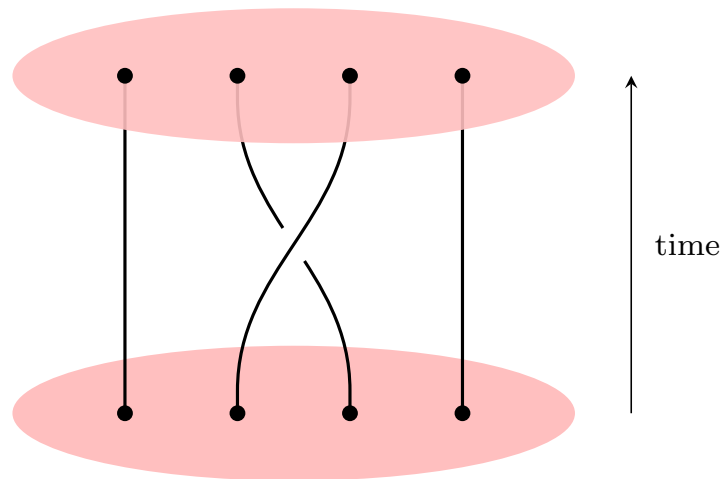
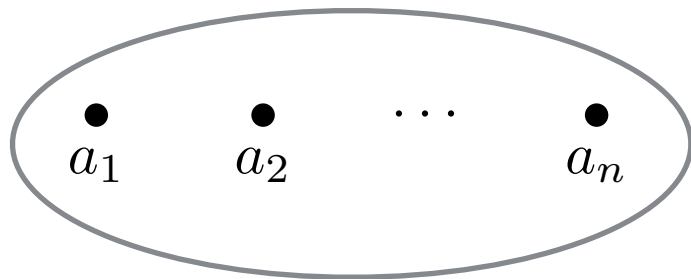
③

(topology)
topological quantum field theory



Recall:

Anyons can be created, braided, and fused.



Goal:

Describe the theory of anyons in “completely” elementary ways with physics motivation.

Algebraic Description of Anyons

An anyon system is characterized by the following data.

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- The anti-particles define an involution $\overline{(\cdot)} : L \rightarrow L$.
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Example: Fibonacci theory Fib

$$L = \{\mathbf{1}, \tau\}, \quad \bar{\tau} = \tau$$

★ Fusion rules

Fusing anyons a and b may result in different anyons $c, d, \dots$

Define $N_{ab}^c = 1$ if c is a possible outcome, and 0 otherwise.

Remark: In general, N_{ab}^c can be > 1 .

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Example: Fib

$$\tau \otimes \tau = \mathbf{1} \oplus \tau$$

★ State spaces

For a collection of anyons $c, a_1, a_2, \dots, a_n$, there are two Hilbert spaces

$V_c^{a_1, \dots, a_n}$: space of physical processes of splitting c into $a_1, \dots, a_n$

$V_{a_1, \dots, a_n}^c$: space of physical processes of fusing $a_1, \dots, a_n$ into c .

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There is a 'time-reverse'/adjoint operator:

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$n = 1$:

$$V_c^a = V_a^c = \delta_{a,c} \mathbb{C} = \delta_{a,c} \text{span} \left\{ \begin{array}{c} | \\ c \end{array} \right\}$$

'identity' process 

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Call $(a, b; c)$ admissible if $N_{ab}^c = 1$. In this case,

$$V_c^{ab} = \text{span} \left\{ \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \\ \diagup \quad \diagdown \\ c \end{array} \right\}$$

splitting process

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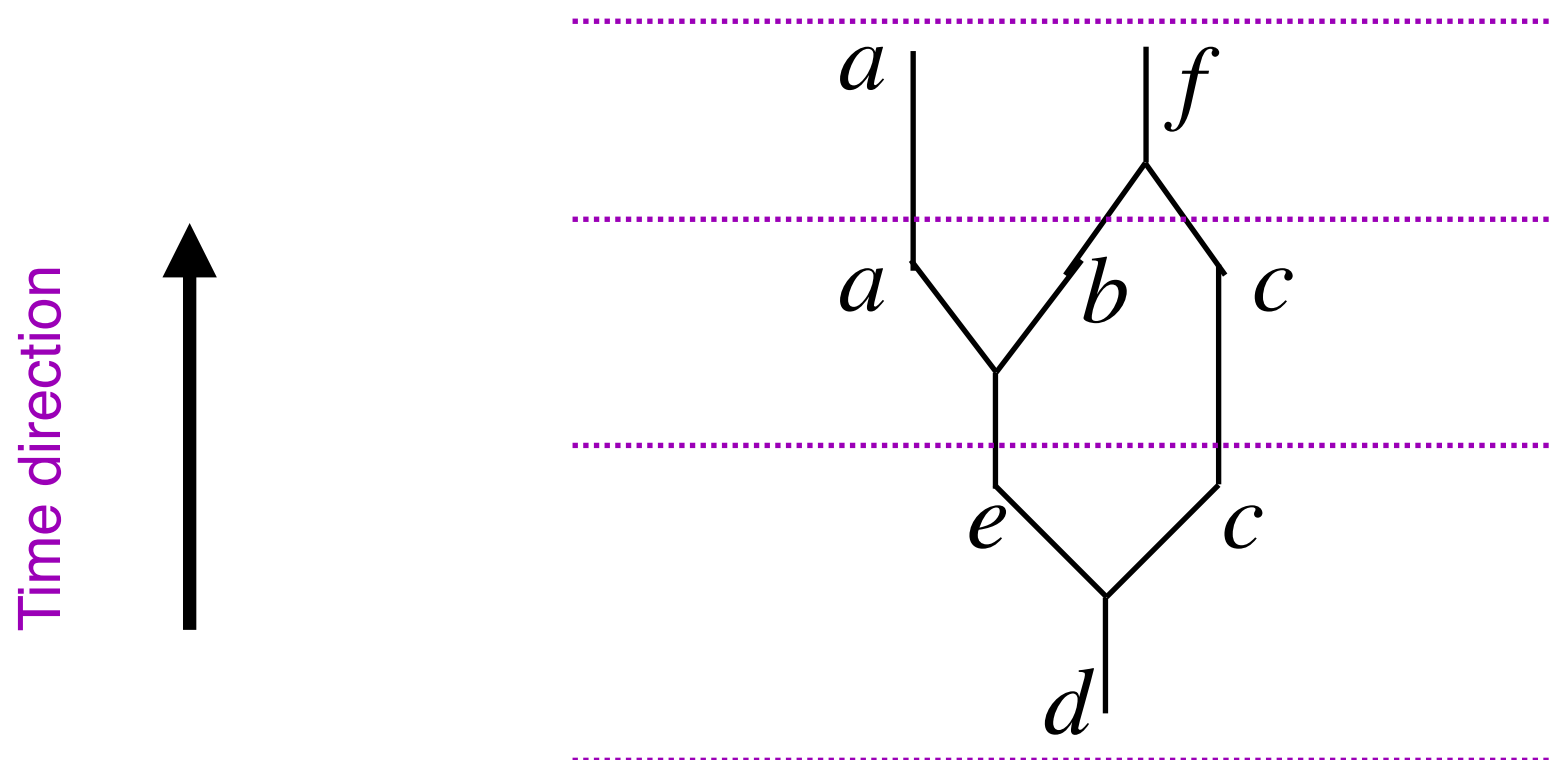
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fusion process

Processes can be “composed” and happen “in parallel”:



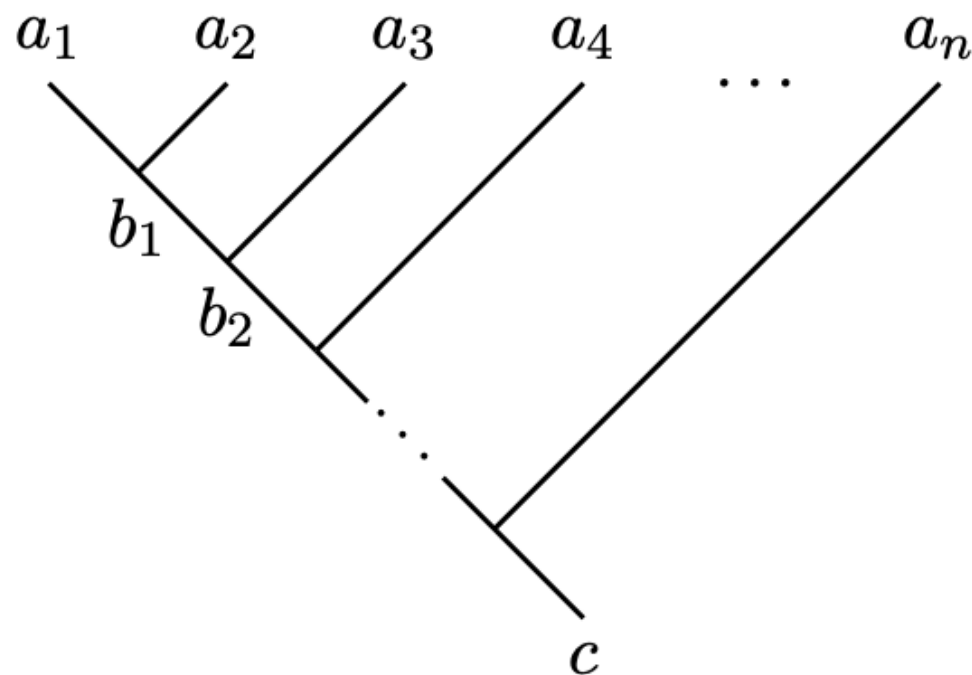
Make the choice so that,

$$\left(\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ c \end{array} \right)^{\dagger} = \begin{array}{c} c \\ | \\ a \quad b \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} c' \\ | \\ a \quad b \\ \diagdown \quad \diagup \\ c \end{array} = \delta_{c,c'} \begin{array}{c} | \\ c \end{array}$$

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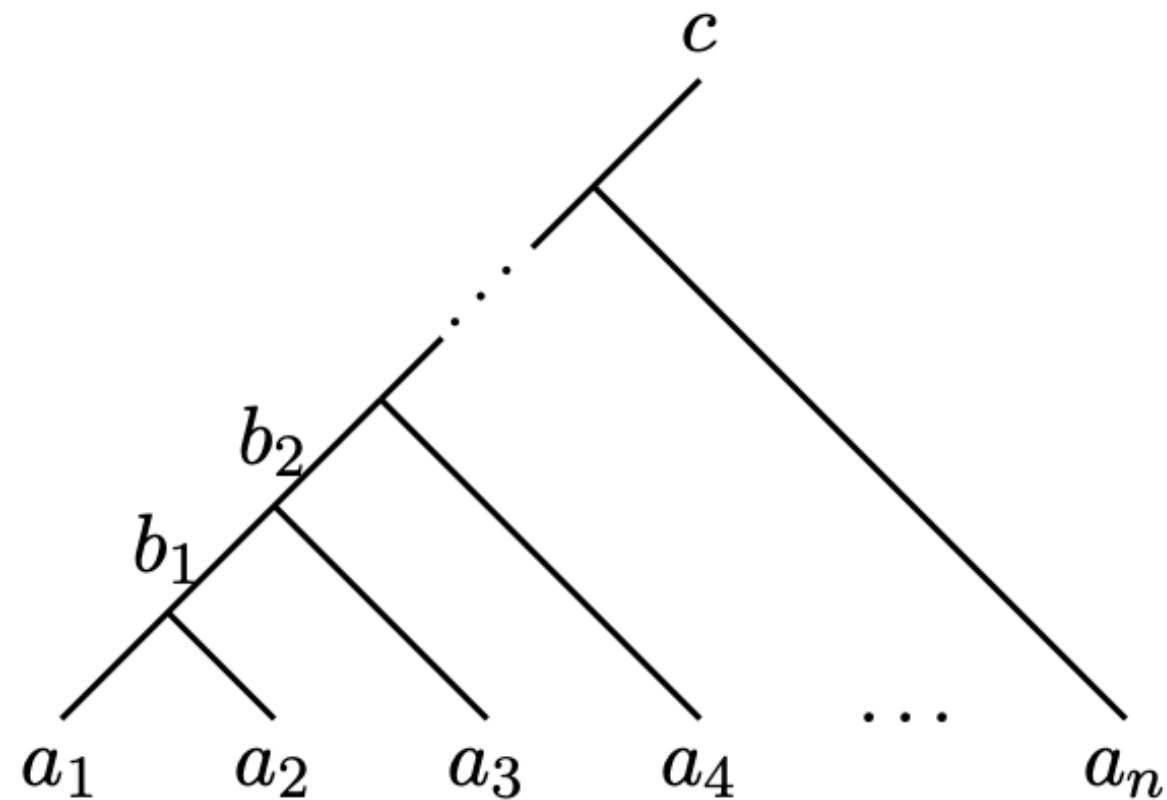
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For general n , an orthonormal basis of $V_c^{a_1, \dots, a_n}$ is given by fixing a binary tree and varying labels of internal edges:



splitting c into b_{n-2} and a_n , followed by splitting b_{n-2} into b_{n-3} and a_{n-1} ,, followed by splitting b_1 into a_1 and a_2

The adjoint gives an orthonormal basis of $V_{a_1, \dots, a_n}^c$:



fusing a_1 and a_2 into b_1 , followed by
fusing b_1 and a_3 into b_2 ,

Relations between different bases:

$$\begin{array}{c} a & b & c \\ & \diagdown & / \\ & x & \\ & \diagup & \diagdown \\ & d & \end{array} = \sum_y F_{d;yx}^{abc} \begin{array}{c} a & b & c \\ & \diagdown & / \\ & y & \\ & \diagup & \diagdown \\ & d & \end{array}$$

F -move relating two bases of V_d^{abc}

Relations between different bases:

The diagram illustrates an F-move, which is a relation between two different bases of a vector space V_d^{abc} . On the left, a tree diagram has three top nodes labeled a , b , and c . Node a is connected to node x , and node b is also connected to node x . Node x is then connected to node d . Node c is connected directly to node d . On the right, another tree diagram has the same three top nodes a , b , and c . Node a is connected to node y , and node b is also connected to node y . Node y is then connected to node d . Node c is connected directly to node d . Between the two diagrams is an equals sign followed by a summation over y of the coefficient $F_{d;yx}^{abc}$.

$$\begin{array}{c} a & b & c \\ & \diagdown & / \\ & x & \\ & \diagup & \diagdown \\ & d & \end{array} = \sum_y F_{d;yx}^{abc} \begin{array}{c} a & b & c \\ & \diagdown & / \\ & y & \\ & \diagup & \diagdown \\ & d & \end{array}$$

F -move relating two bases of V_d^{abc}

Possible x -values $x_1, \dots, x_k$; possible y -values $y_1, \dots, y_k$. Define the matrix F_d^{abc} with $(F_d^{abc})_{ij} = F_{d;y_i x_j}^{abc}$. It is unitary.

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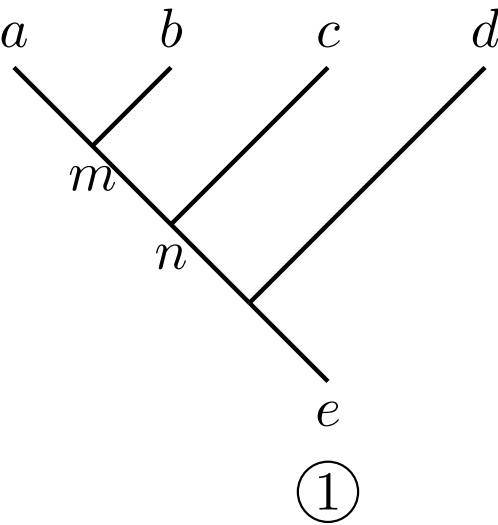
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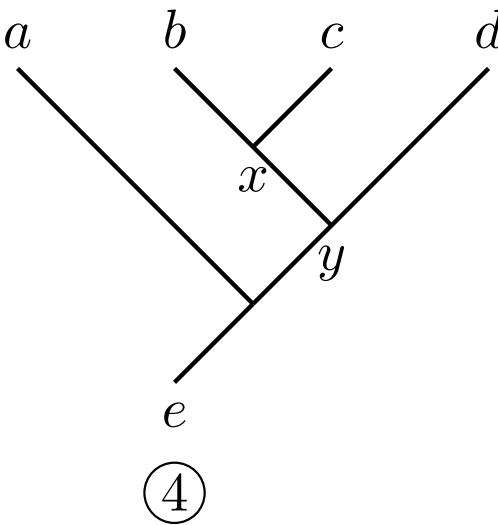
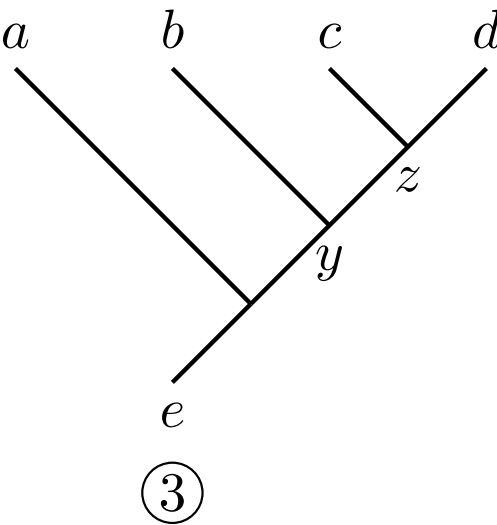
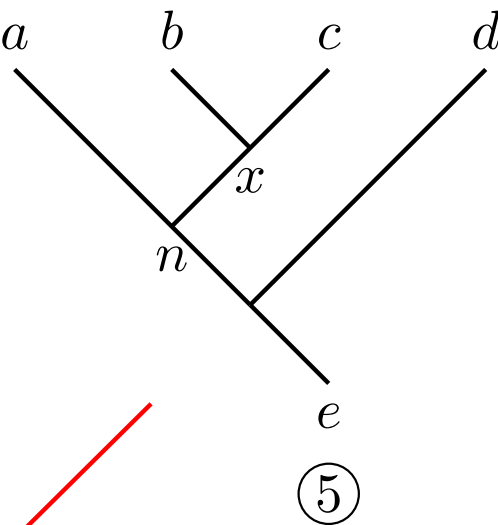
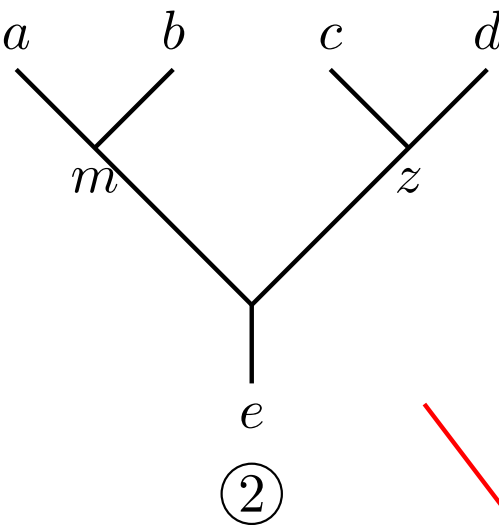
Example: Fib

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \quad \phi = \frac{1 + \sqrt{5}}{2}$$

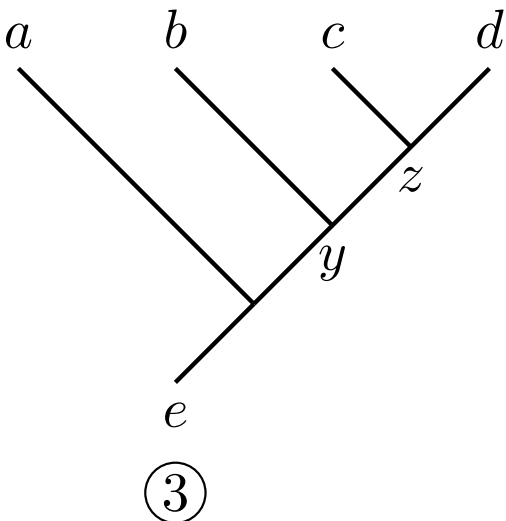
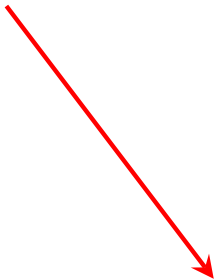
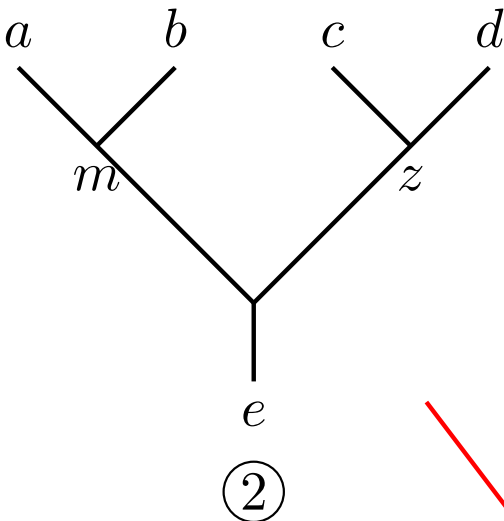
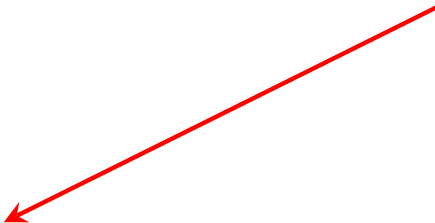
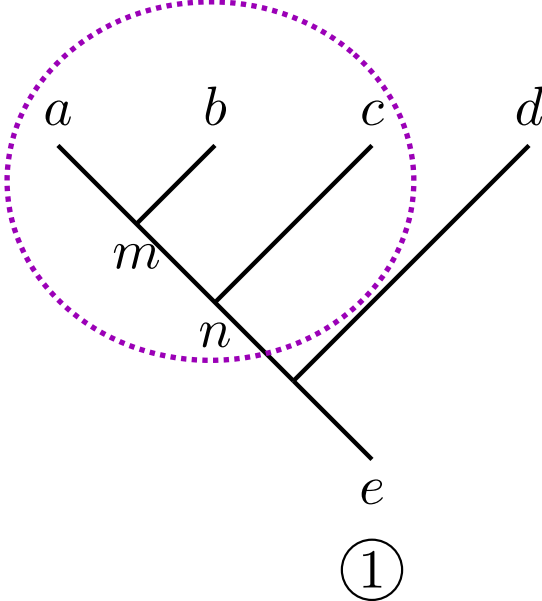
Consistency:



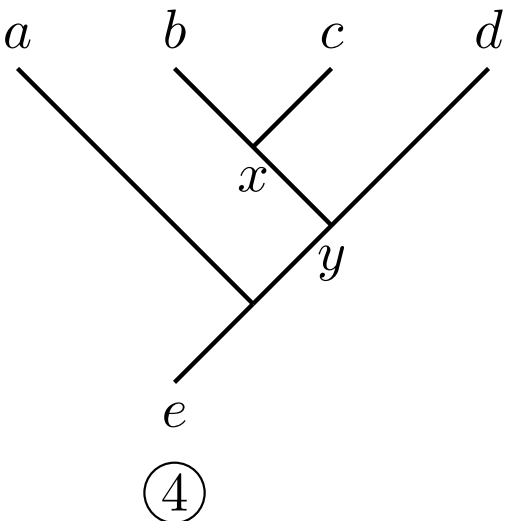
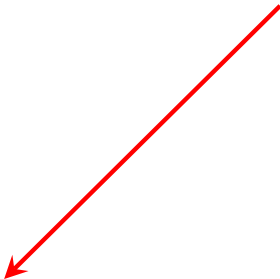
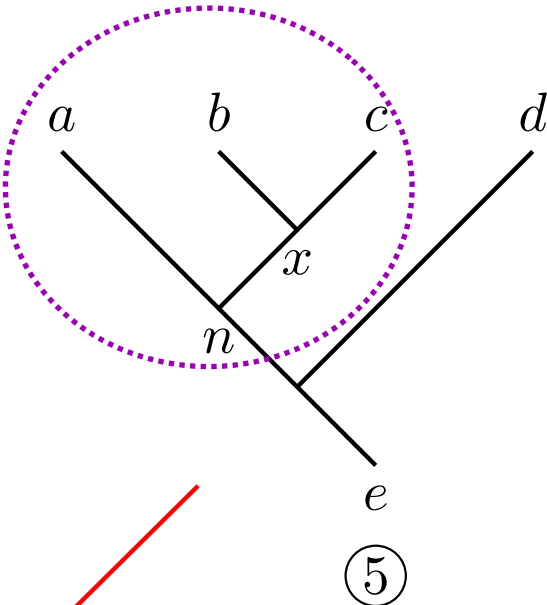
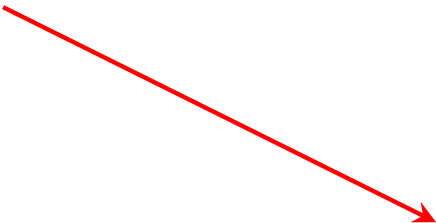
Pentagon Equation



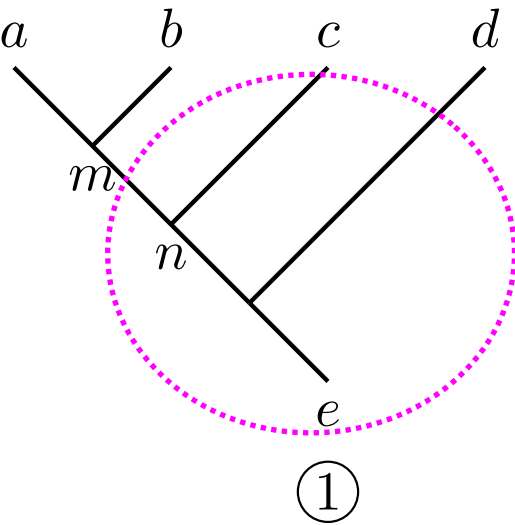
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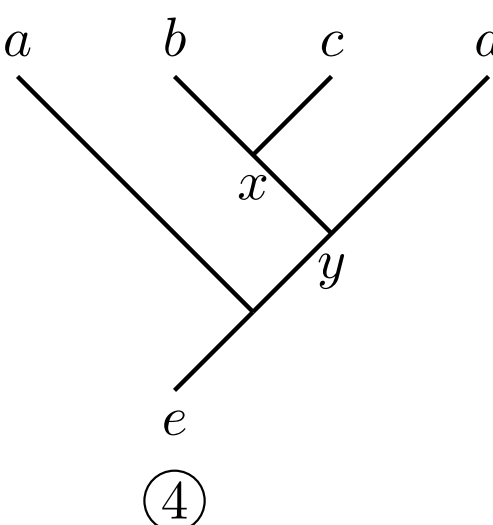
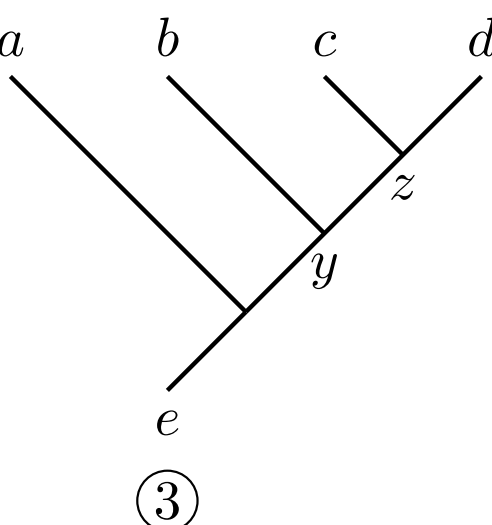
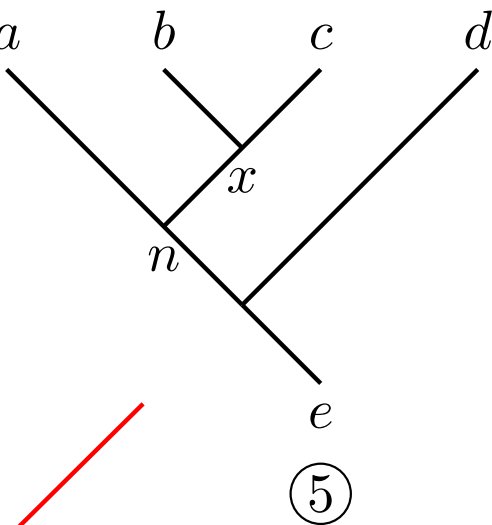
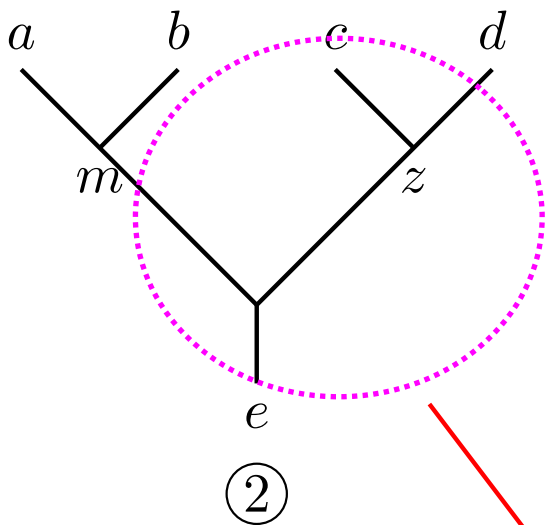
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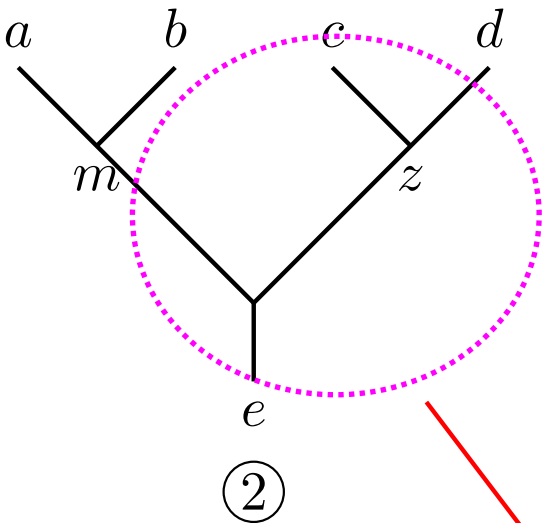
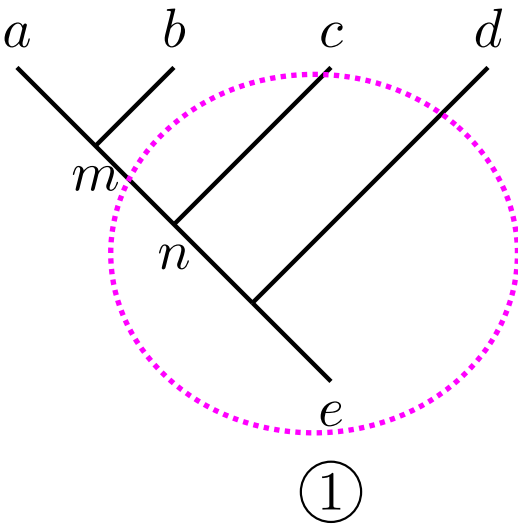


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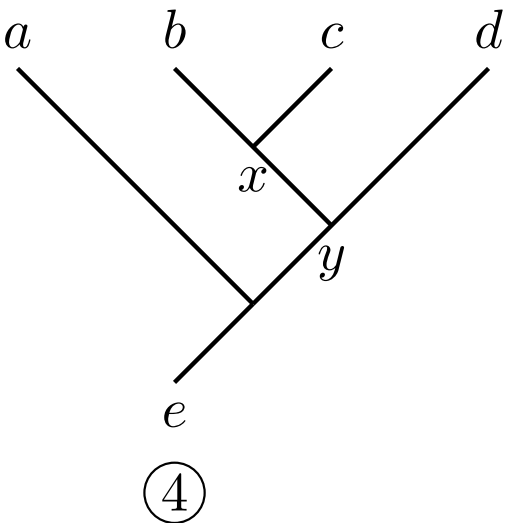
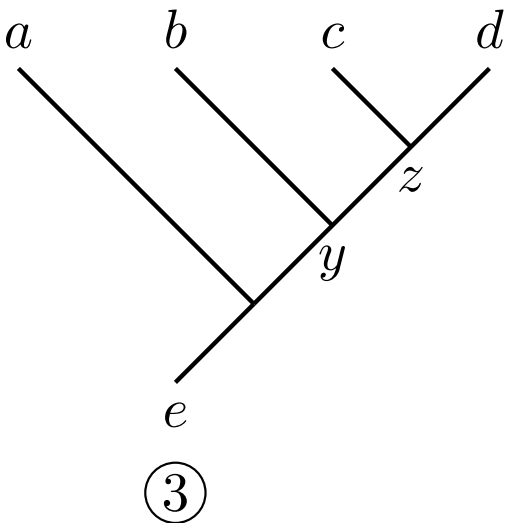
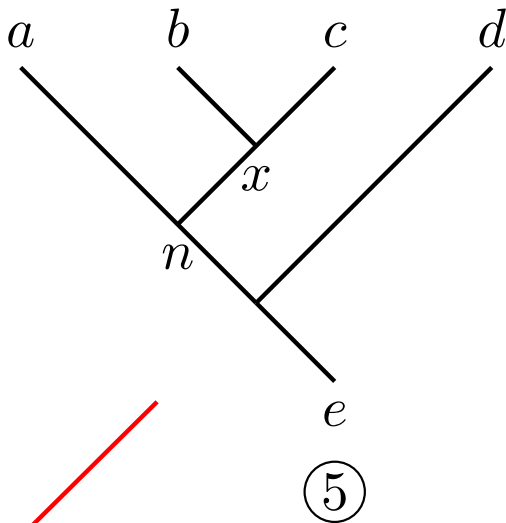
Consistency:

Pentagon Equation



$$F_{e;zn}^{mcd} F_{e;ym}^{abz} = \sum_{x \in L} F_{n;xm}^{abc} F_{e;yn}^{axd} F_{y;zx}^{bcd}$$

$$\forall a, b, c, d, e, m, n, y, z$$



Pentagon equations guarantee consistency of basis change regardless of the particular sequence of F-moves.

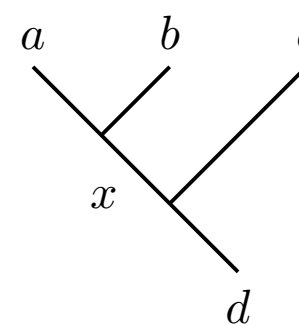
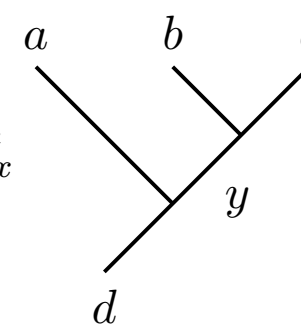
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shorthand
 $\Rightarrow$
 notation

$$(\dots) \cdot \xrightarrow{F} \cdot (\dots)$$

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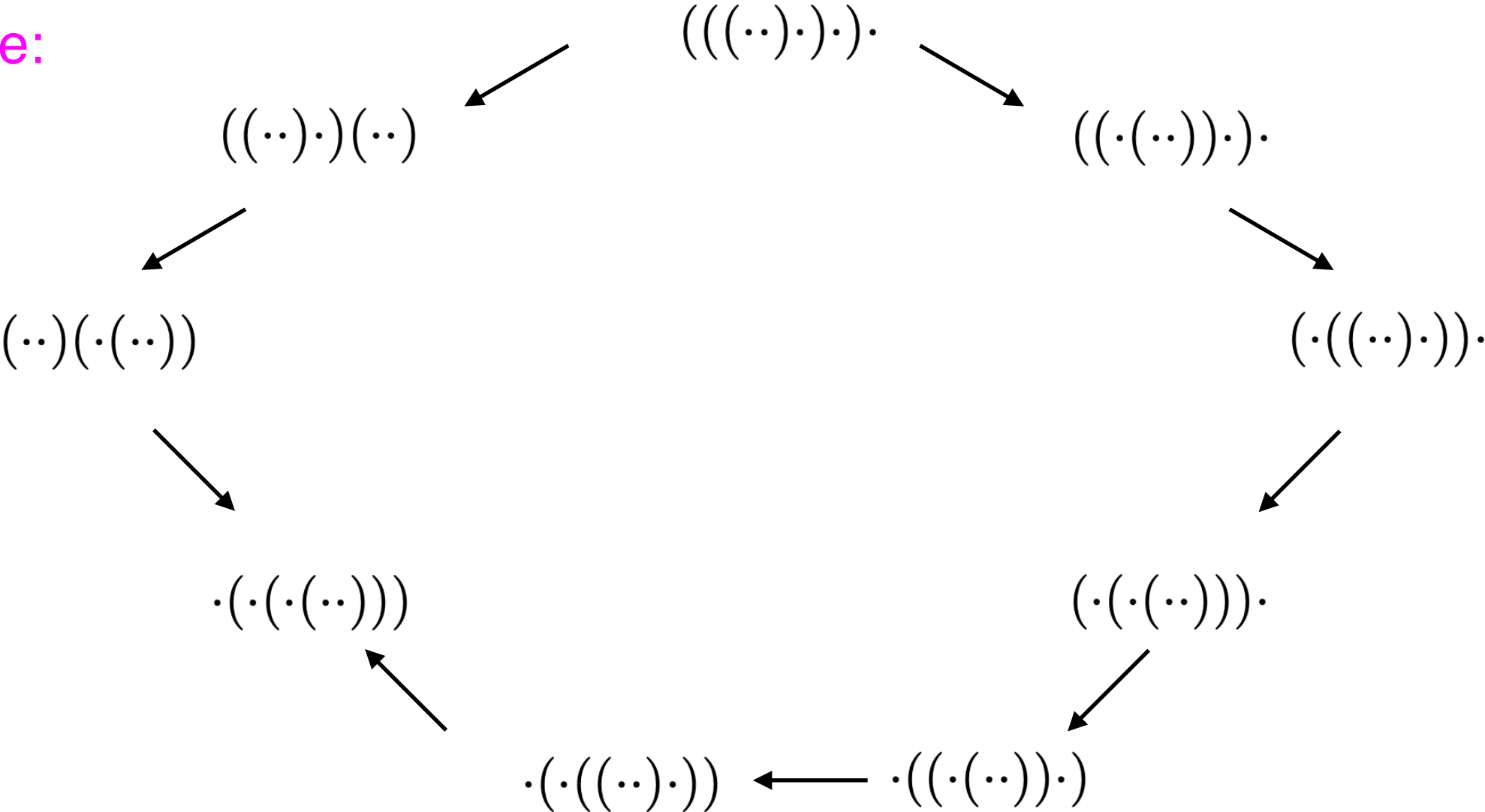
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Exercise:



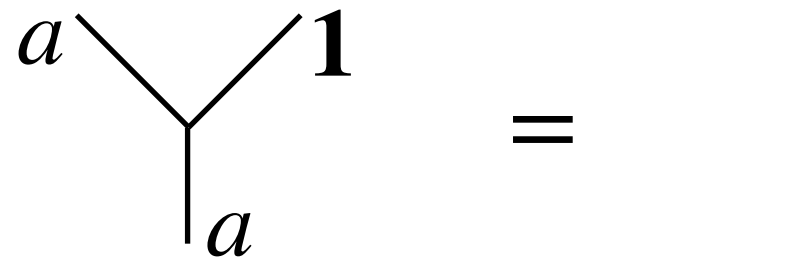
The special role of **1**:

$V_c^{a_1, \dots, a_{i-1}, a_i, \dots, a_n}$ and $V_c^{a_1, \dots, a_{i-1}, \mathbf{1}, a_i, \dots, a_n}$ are naturally identified.

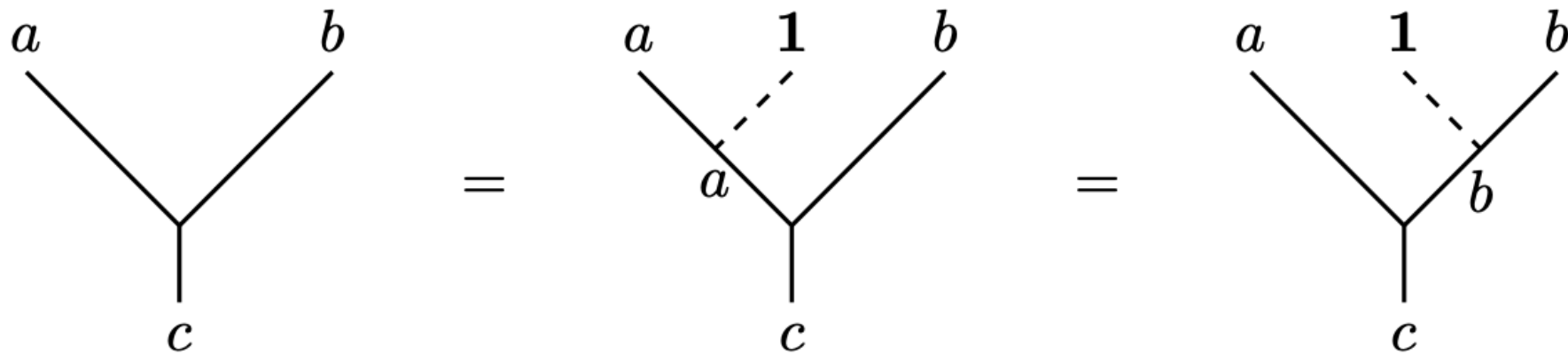
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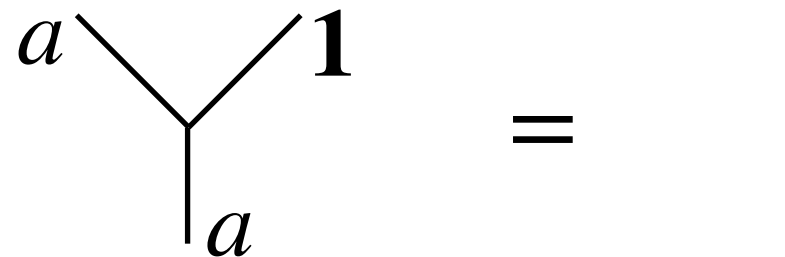
An edge labelled by **1** can be erased or inserted



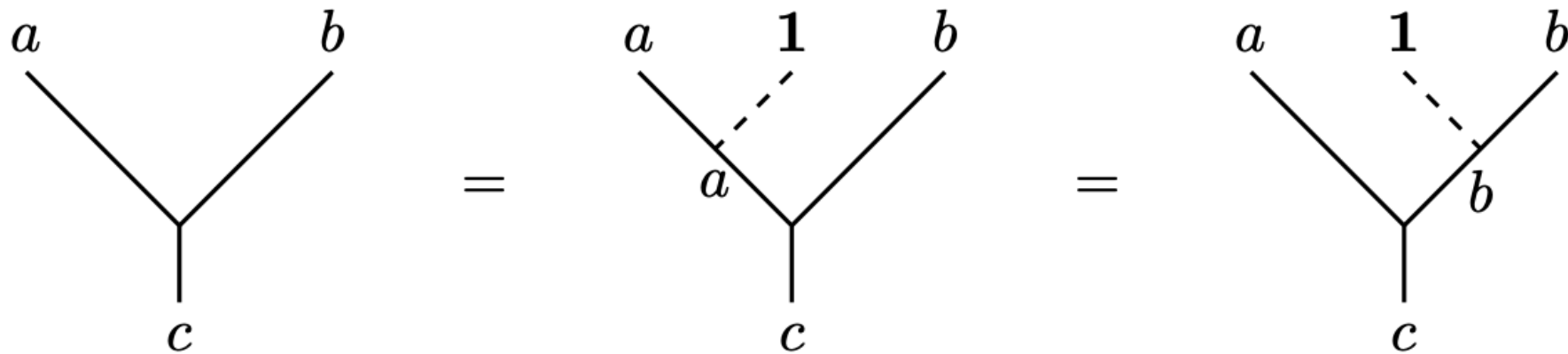
$$\Rightarrow F_{c;ba}^{a\mathbf{1}b} = 1$$

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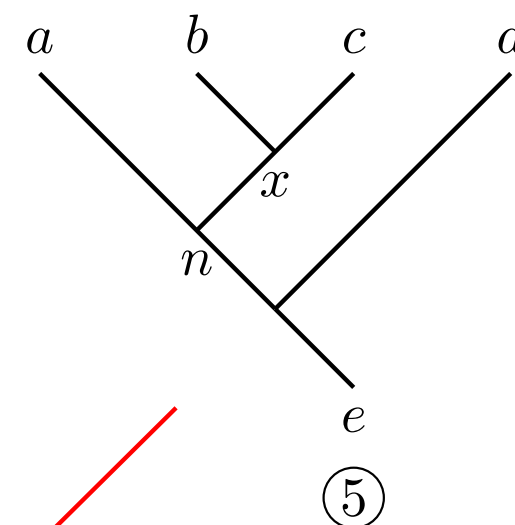
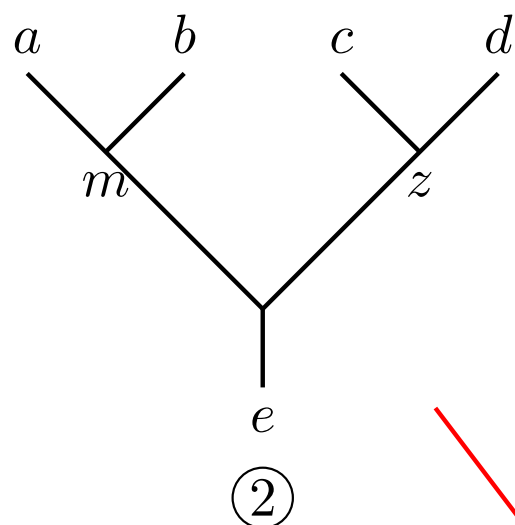
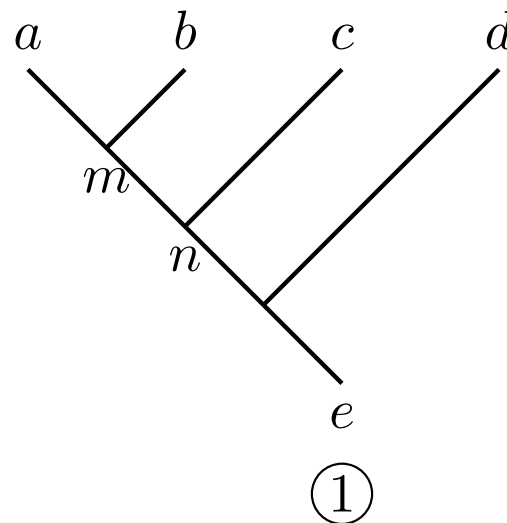
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$$\Rightarrow F_{c;ba}^{a\mathbf{1}b} = 1$$

In fact, it follows that $F_d^{abc} = (1)$ whenever one of a, b, c is **1**

set $a = b = \mathbf{1}$, then
 $m = \mathbf{1}, n = x = c,$
 $y = z = e$

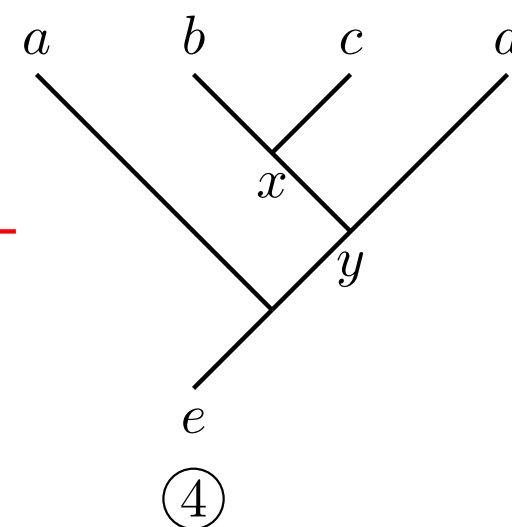
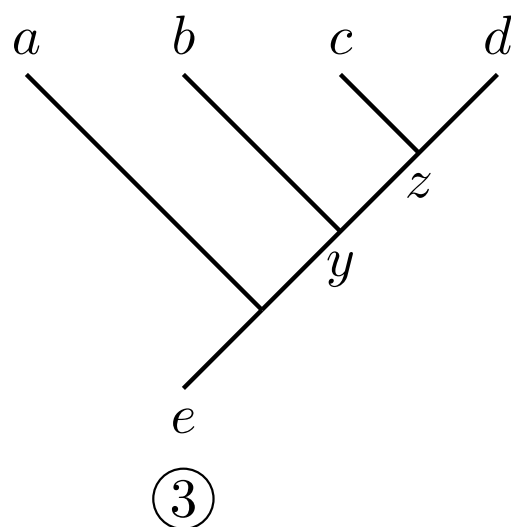


Pentagon Equation

$$F_{e;ec}^{1cd} = (F_{e;ec}^{1cd})^2$$

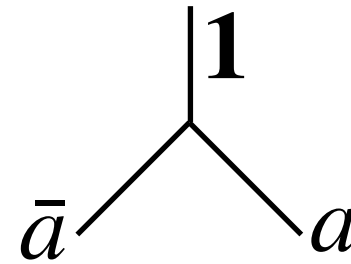
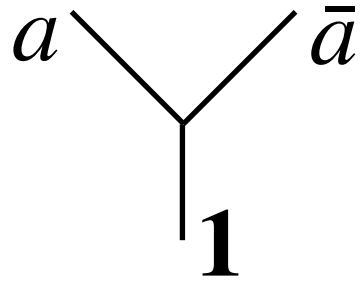


$$F_{e;ec}^{1cd} = 1$$



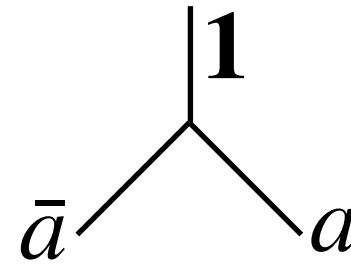
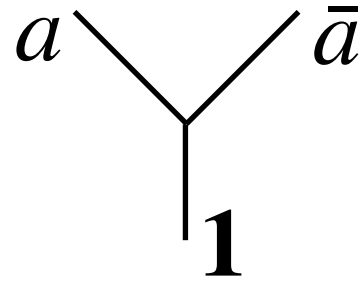
★ Creation/annihilation

Creating/fusing a pair of dual anyons from/to the vacuum:

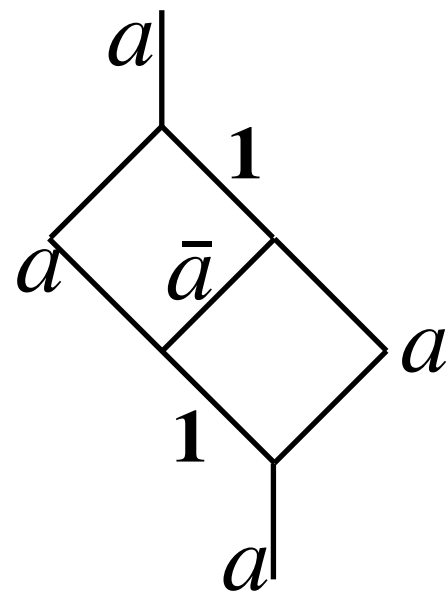


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Creating/fusing a pair of dual anyons from/to the vacuum:

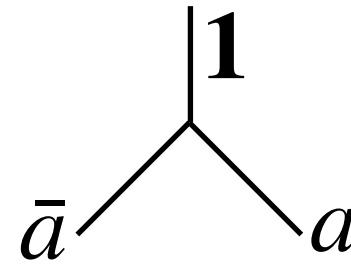
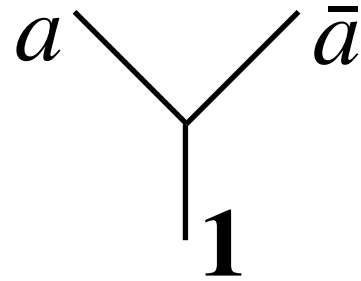


Consider:

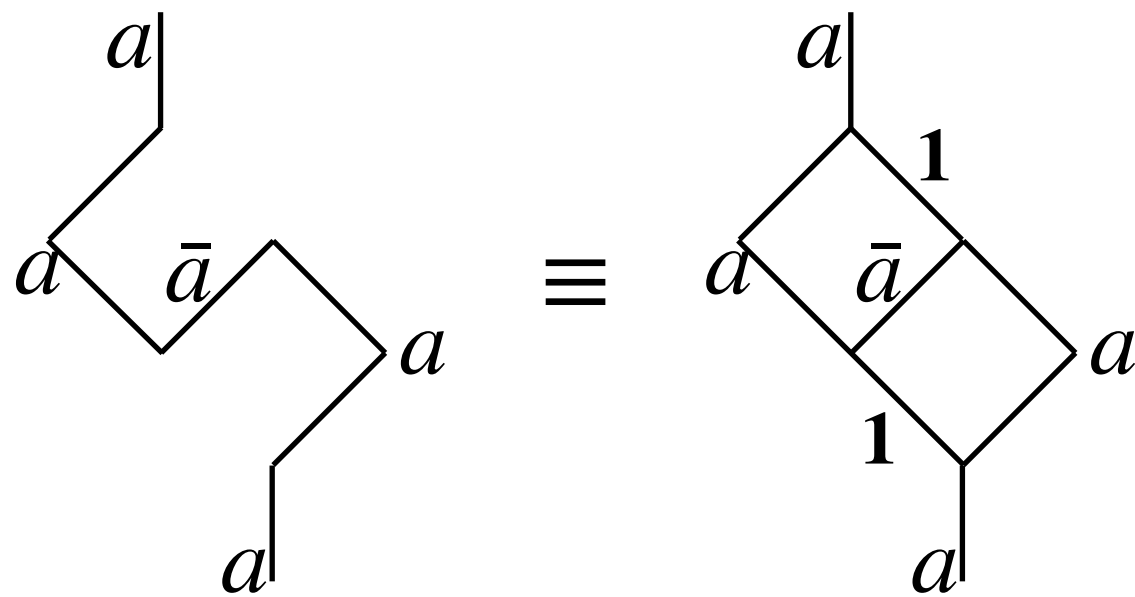


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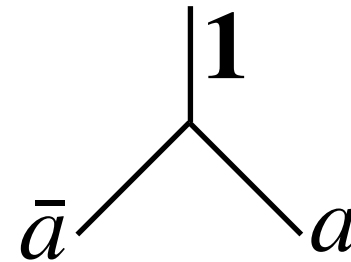
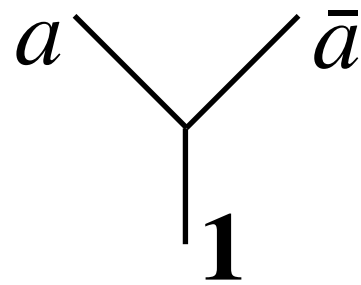


Consider:

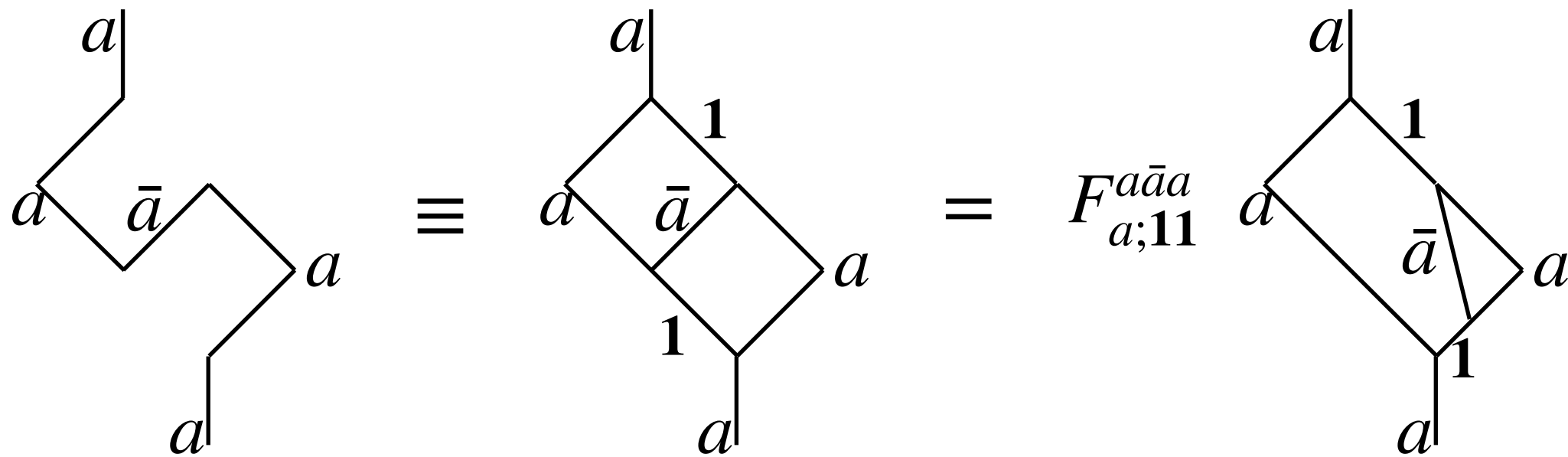


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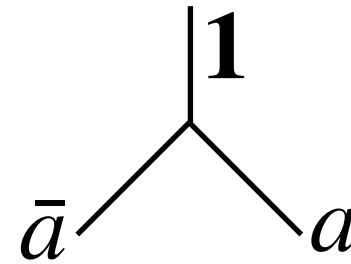
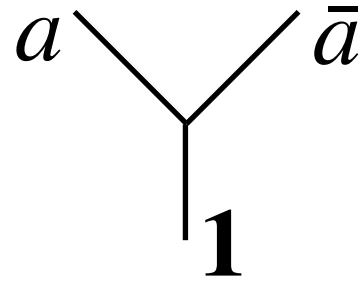


Consider:



★ Creation/annihilation

Creating/fusing a pair of dual anyons from/to the vacuum:



Consider:

$$\begin{array}{c} a \\ | \\ \diagdown \quad \diagup \\ a \quad \bar{a} \\ \diagup \quad \diagdown \\ \quad \quad a \\ | \\ a \end{array} \equiv \begin{array}{c} a \\ | \\ \diagdown \quad \diagup \quad 1 \\ a \quad \bar{a} \\ \diagup \quad \diagdown \\ \quad \quad 1 \\ | \\ a \end{array} = F_{a;11}^{a\bar{a}a} \begin{array}{c} a \\ | \\ \diagdown \quad \diagup \quad 1 \\ a \quad \bar{a} \\ \diagup \quad \diagdown \\ \quad \quad 1 \\ | \\ a \end{array} = F_{a;11}^{a\bar{a}a} \begin{array}{c} | \\ \\ \\ \\ a \end{array}$$

Similarly,

$$\begin{array}{ccccccc}
 \begin{array}{c} \bar{a} \\ \diagup \\ \bar{a} \end{array} & \begin{array}{c} \bar{a} \\ \diagdown \\ \bar{a} \end{array} & = & \begin{array}{c} \bar{a} \\ \diagup \\ \bar{a} \end{array} & \begin{array}{c} \bar{a} \\ \diagdown \\ \bar{a} \end{array} & = & \overline{F_{\bar{a};11}^{\bar{a}a\bar{a}}} \begin{array}{c} \bar{a} \\ \diagup \\ \bar{a} \end{array} & \begin{array}{c} \bar{a} \\ \diagdown \\ \bar{a} \end{array} & = & \overline{F_{\bar{a};11}^{\bar{a}a\bar{a}}} \begin{array}{c} \bar{a} \\ \diagup \\ \bar{a} \end{array} & \begin{array}{c} \bar{a} \\ \diagdown \\ \bar{a} \end{array}
 \end{array}$$

Similarly,

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = \overline{F_{\bar{a};\mathbf{11}}^{a\bar{a}a\bar{a}}} \text{Diagram 3} = \overline{F_{\bar{a};\mathbf{11}}^{a\bar{a}a\bar{a}}} \text{Diagram 4}
 \end{aligned}$$

Renormalize creation/fusion pair to have homotopy invariance

$$\Rightarrow F_{a;\mathbf{11}}^{a\bar{a}a} = \overline{F_{\bar{a};\mathbf{11}}^{a\bar{a}a\bar{a}}} \neq 0 \quad \text{Let } \phi_a := (F_{a;\mathbf{11}}^{a\bar{a}a})^{-\frac{1}{2}}$$

Similarly,

$$\begin{array}{c} \bar{a} \\ \diagup \\ a \\ \diagdown \\ \bar{a} \end{array} = \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \quad 1 \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \quad 1 \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ | \\ \bar{a} \end{array}$$

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$$\begin{array}{c} \curvearrowright \\ a \end{array} := \phi_a \begin{array}{c} a \\ \diagdown \quad \diagup \\ \bar{a} \end{array}$$

Similarly,

$$\begin{array}{c} \bar{a} \\ \diagup \\ a \\ \diagdown \\ \bar{a} \end{array} = \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ | \end{array}$$

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Similarly,

$$\begin{array}{c} \bar{a} \\ \diagup \\ a \\ \diagdown \\ \bar{a} \end{array} = \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \quad 1 \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ \diagup \quad \diagdown \\ a \quad 1 \\ \diagdown \quad \diagup \\ \bar{a} \end{array} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \begin{array}{c} \bar{a} \\ | \\ \bar{a} \end{array}$$

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Renormalize creation/fusion pair to have homotopy invariance

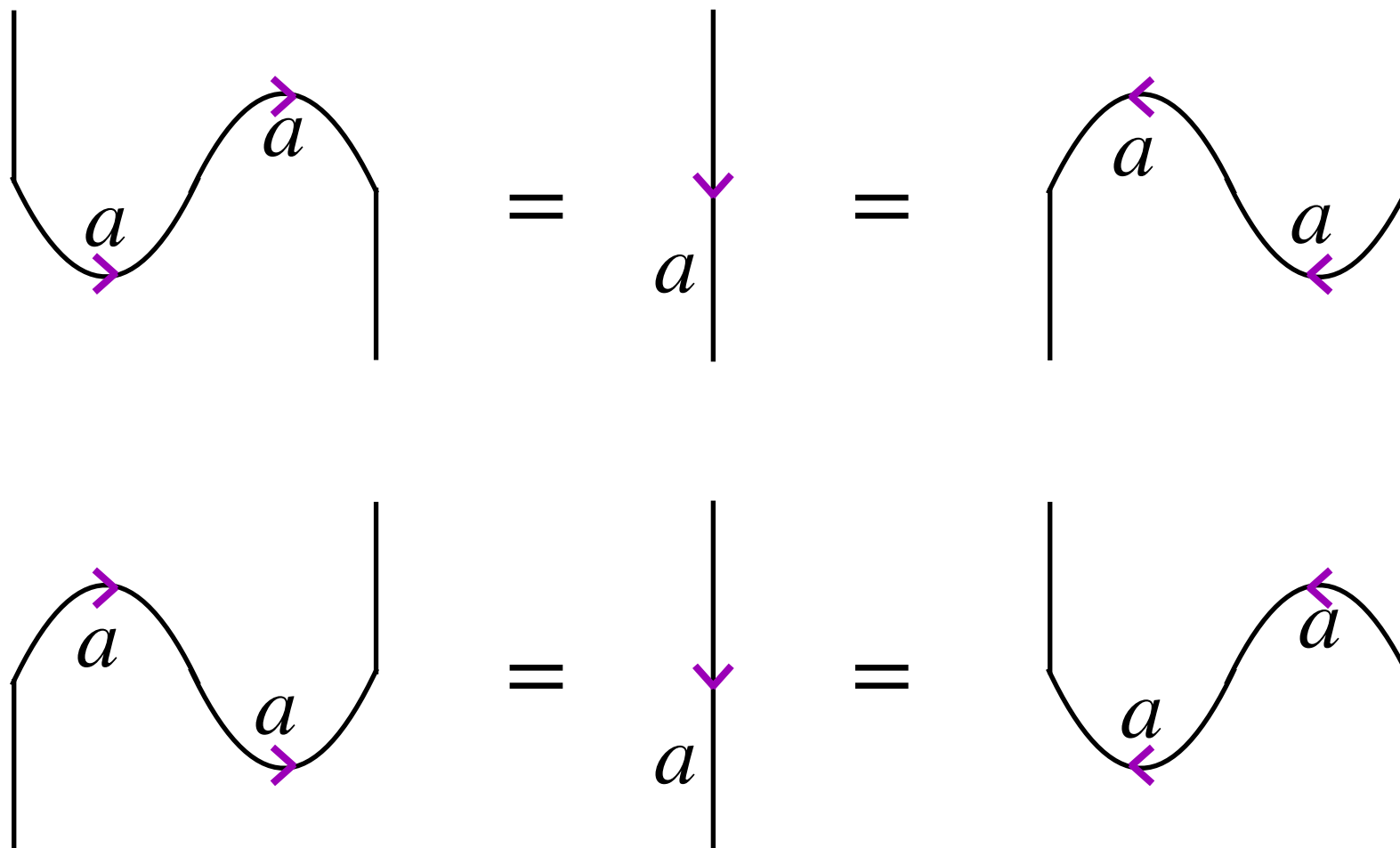
$$\Rightarrow F_{a;11}^{a\bar{a}a} = \overline{F_{\bar{a};11}^{a\bar{a}a}} \neq 0 \quad \text{Let } \phi_a := (F_{a;11}^{a\bar{a}a})^{-\frac{1}{2}}$$

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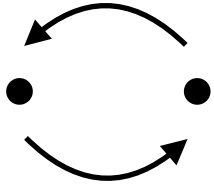


$$\begin{array}{ccccc}
 \begin{array}{c} \text{Diagram 1: A vertical line on the left and right, connected by a curve. The curve has a minimum at the bottom and a maximum at the top. Two purple arrows point to the minimum and maximum, both labeled 'a'.} \end{array} & = & \begin{array}{c} \text{Diagram 2: A vertical line with a purple arrow pointing down, labeled 'a'.} \end{array} & = & \begin{array}{c} \text{Diagram 3: A vertical line on the left and right, connected by a curve. The curve has a maximum at the top and a minimum at the bottom. Two purple arrows point to the maximum and minimum, both labeled 'a'.} \end{array} \\
 \\
 \begin{array}{c} \text{Diagram 4: A vertical line on the left and right, connected by a curve. The curve has a maximum at the top and a minimum at the bottom. Two purple arrows point to the maximum and minimum, both labeled 'a'.} \end{array} & = & \begin{array}{c} \text{Diagram 5: A vertical line with a purple arrow pointing down, labeled 'a'.} \end{array} & = & \begin{array}{c} \text{Diagram 6: A vertical line on the left and right, connected by a curve. The curve has a minimum at the bottom and a maximum at the top. Two purple arrows point to the minimum and maximum, both labeled 'a'.} \end{array}
 \end{array}$$

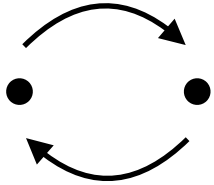
Moreover,

$$\begin{array}{ccccc}
 \begin{array}{c} \text{Diagram 7: A diamond shape with two purple arrows pointing to the top and bottom vertices, both labeled 'a'.} \end{array} & = & |\phi_a|^2 a \begin{array}{c} \text{Diagram 8: A diamond shape with two vertices labeled 'a' and '\bar{a}'.} \end{array} & = & |F_{a;11}^{a\bar{a}a}|^{-1} = \begin{array}{c} \text{Diagram 9: A diamond shape with two purple arrows pointing to the top and bottom vertices, both labeled 'a'.} \end{array} \\
 & & \text{quantum dimension} & & \\
 & & \text{dim}_q a & &
 \end{array}$$

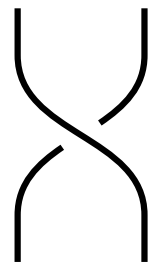
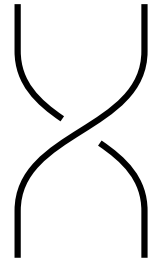
★ Braiding



counterclockwise

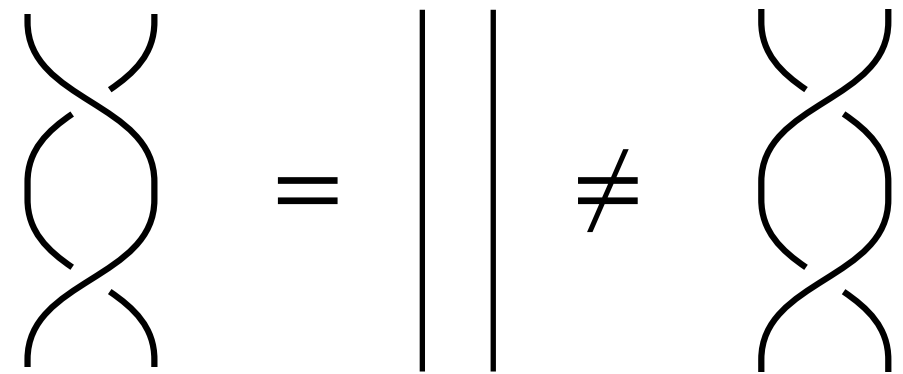
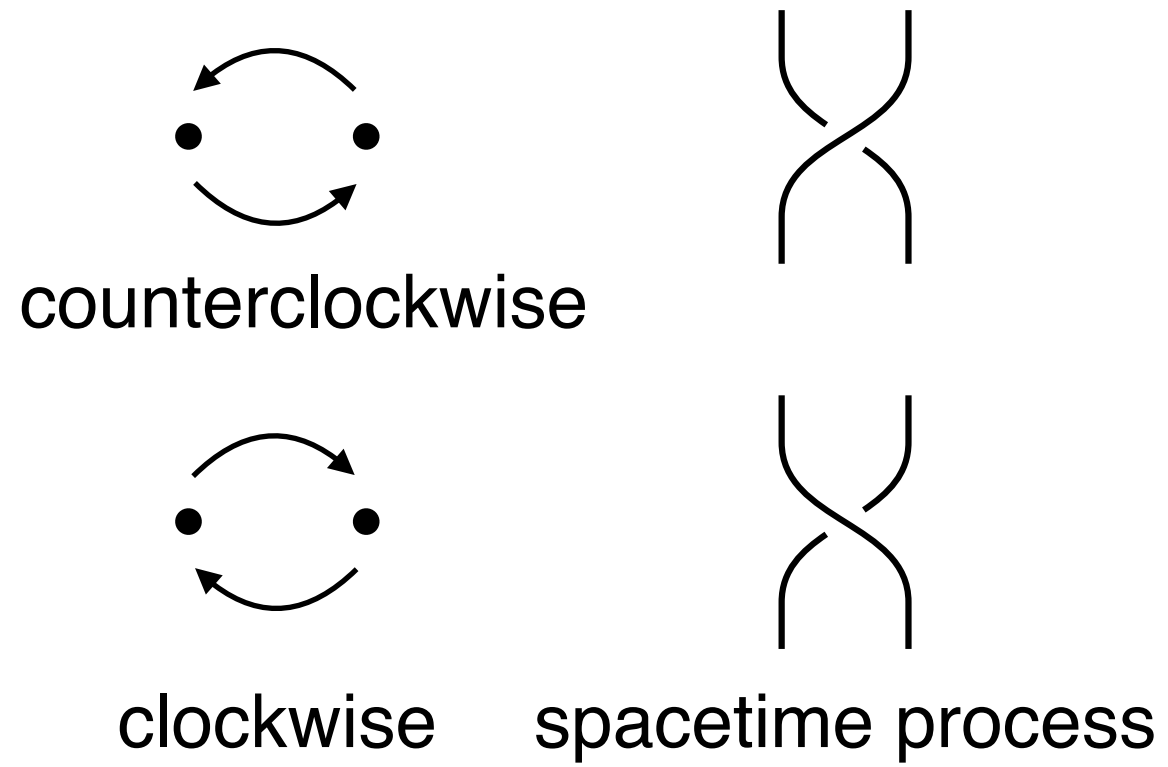


clockwise

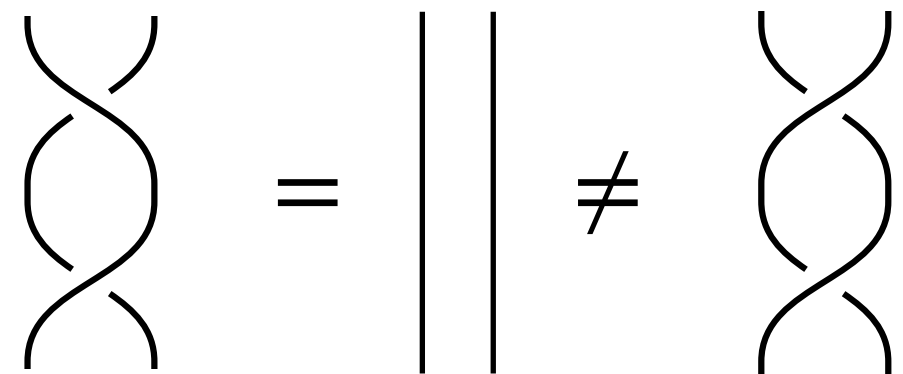
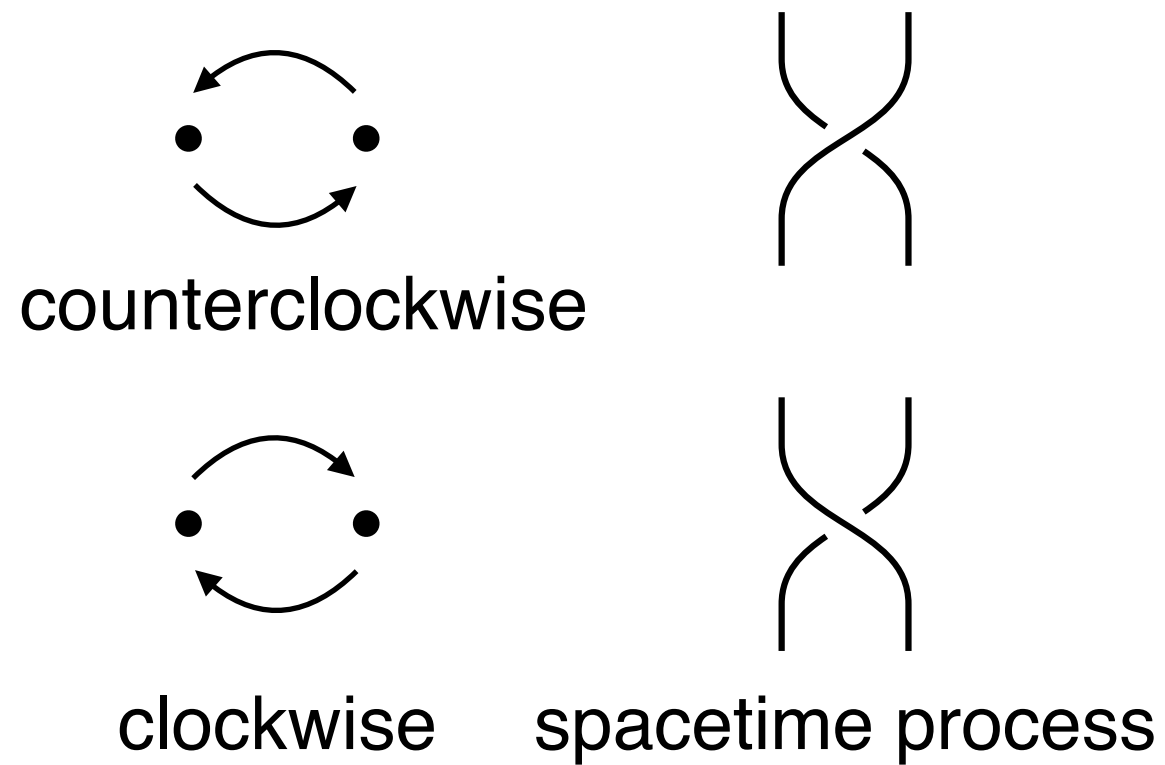


spacetime process

★ Braiding

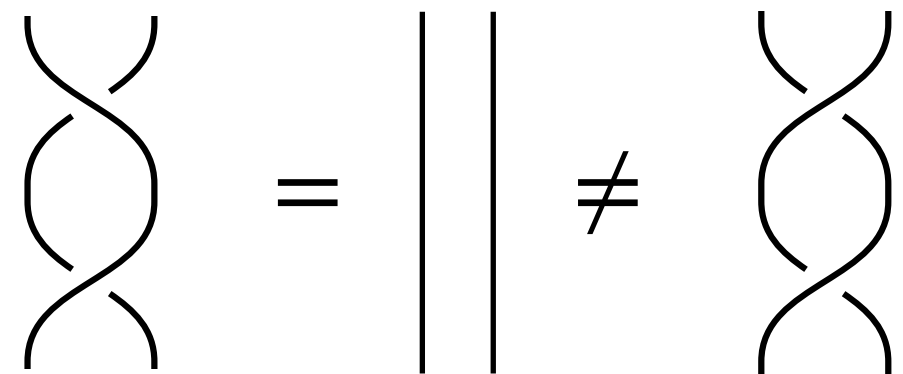
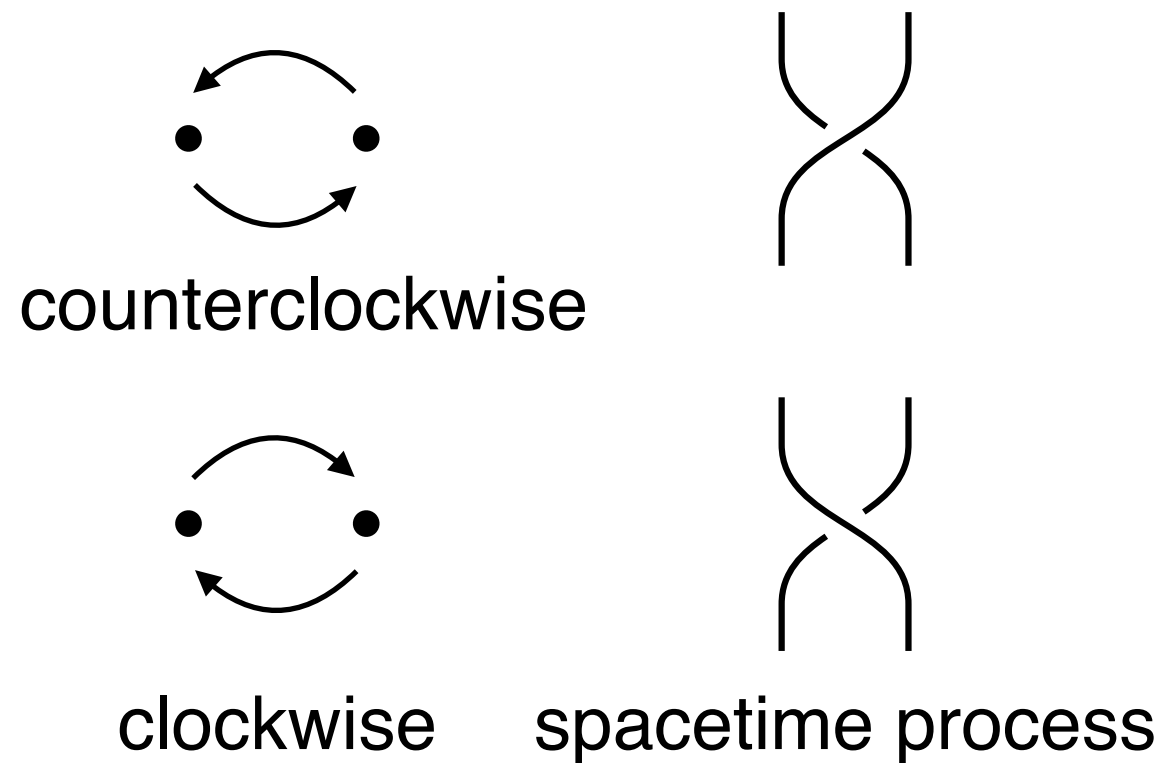


★ Braiding

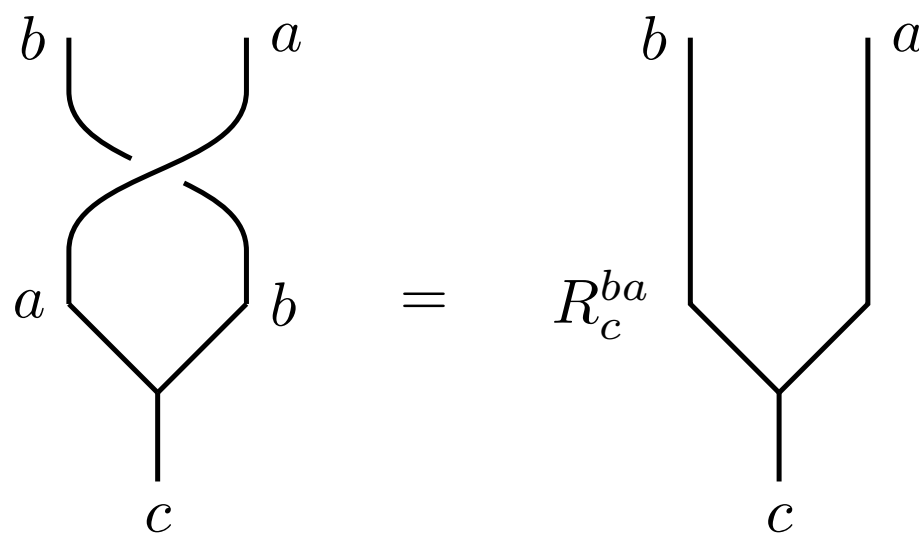


Braiding induces unitary maps: $V_c^{a_1, \dots, a_{i-1}, a_i, \dots, a_n} \longrightarrow V_c^{a_1, \dots, a_i, a_{i-1}, \dots, a_n}$

★ Braiding



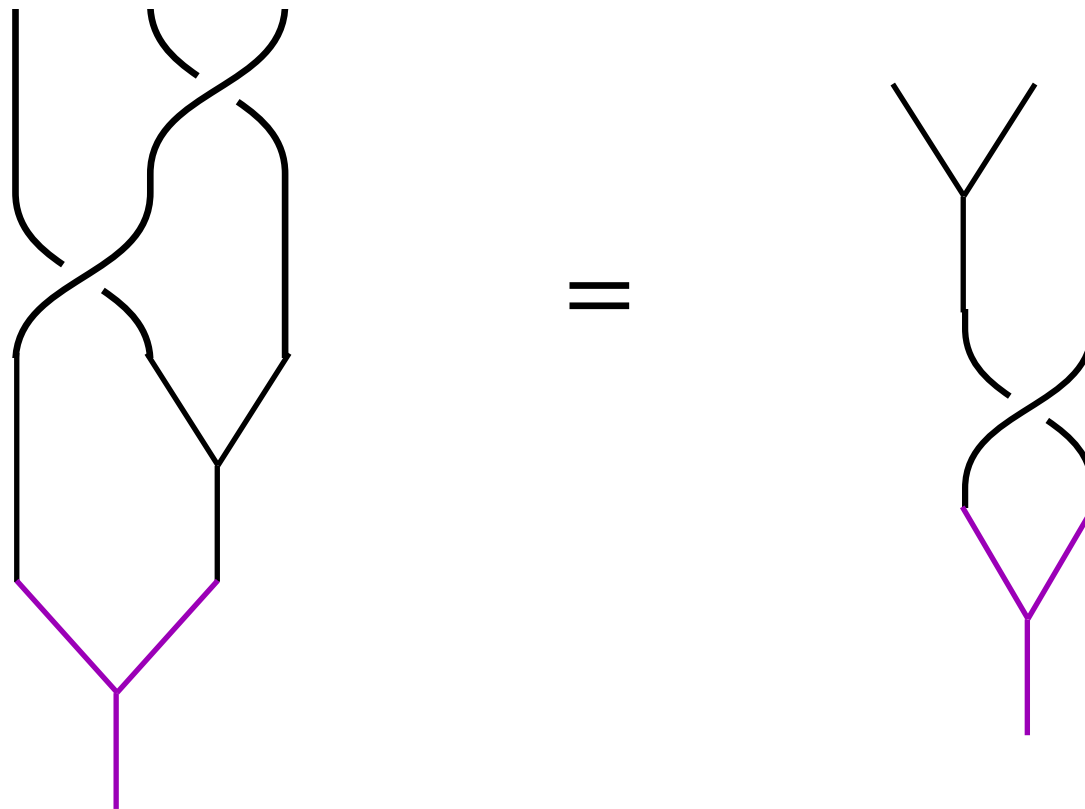
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R-move

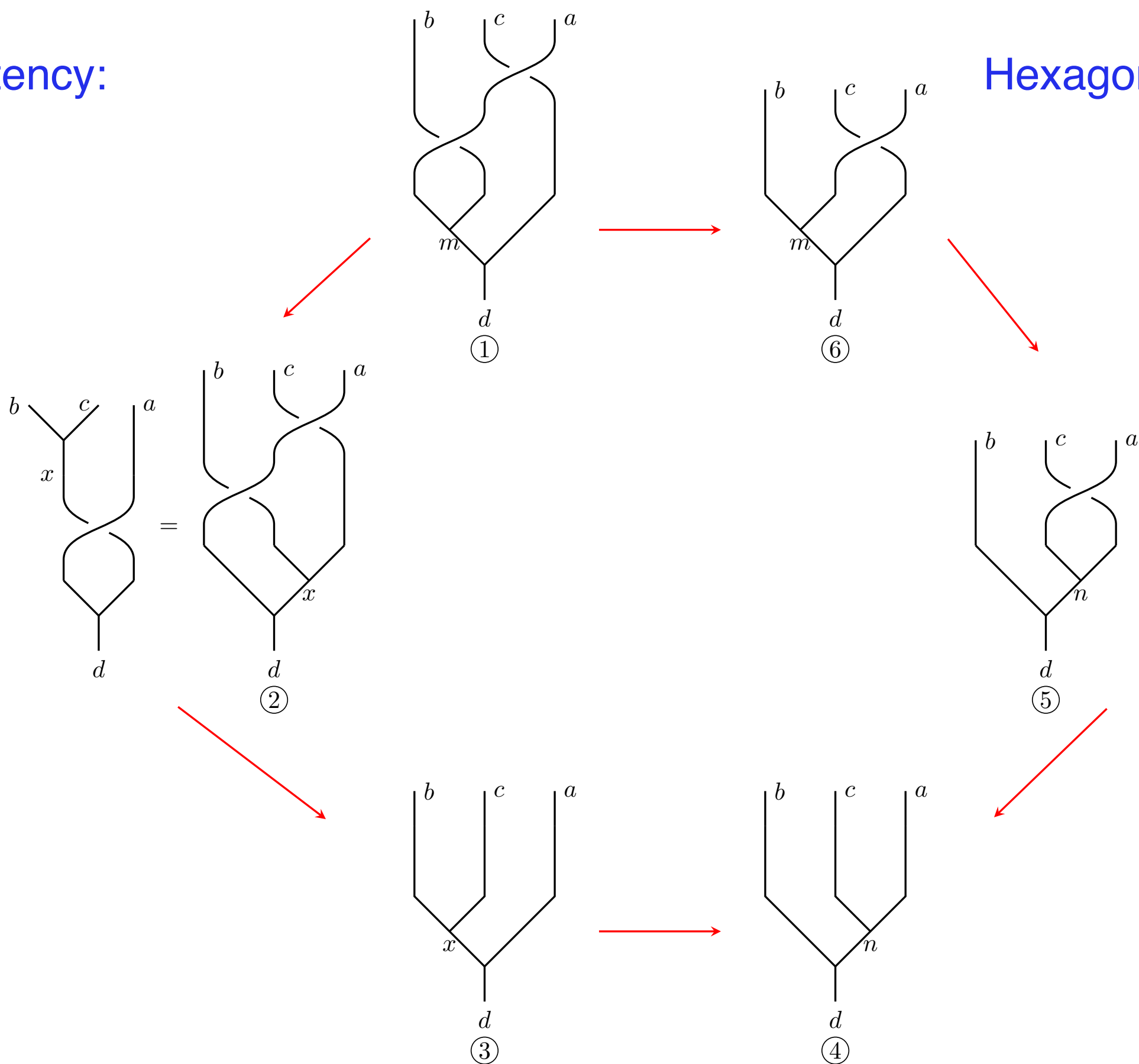
R -symbols are phases since they are unitary

Naturality: braiding commutes with physical processes



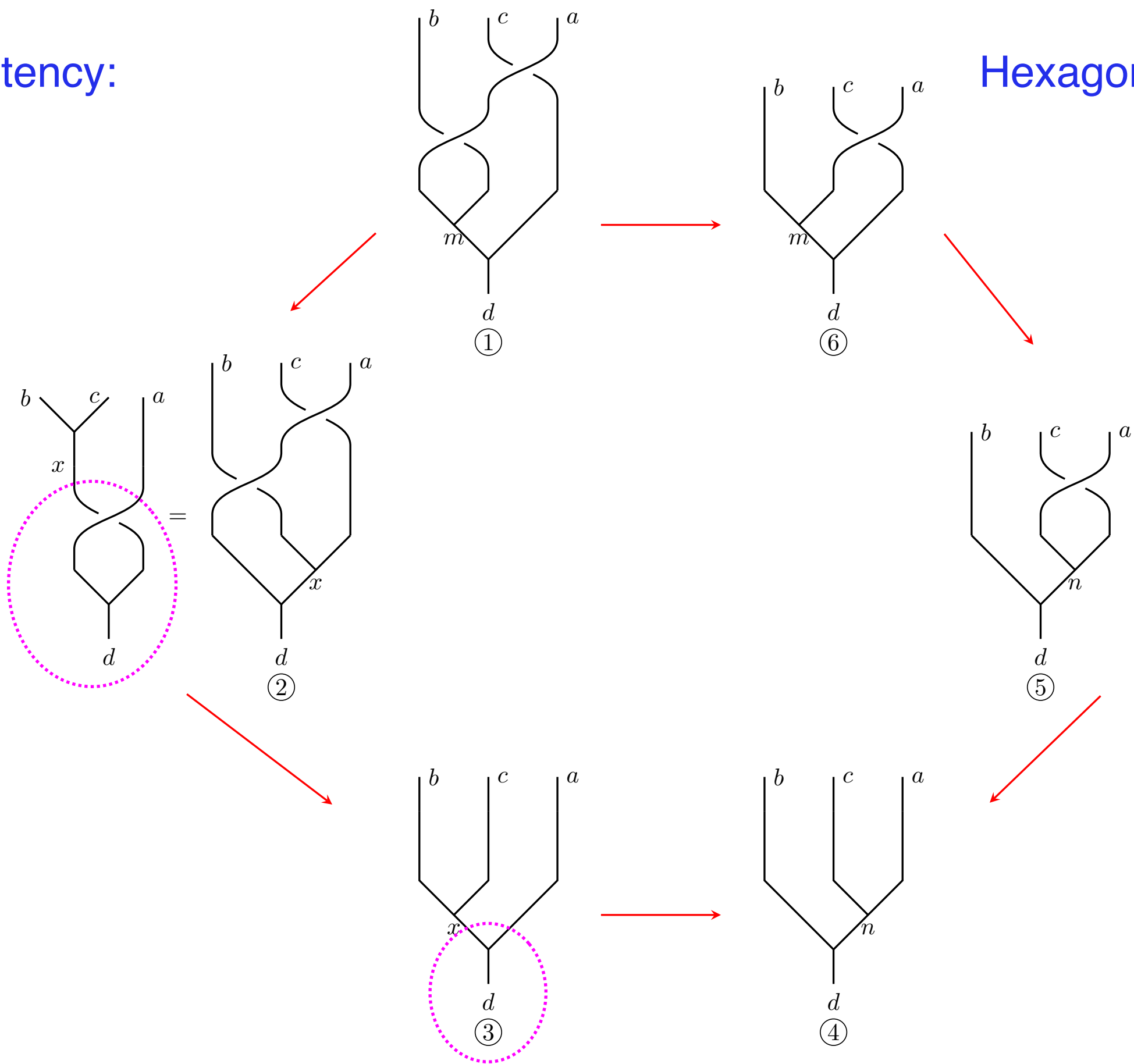
Consistency:

Hexagon Equation 1



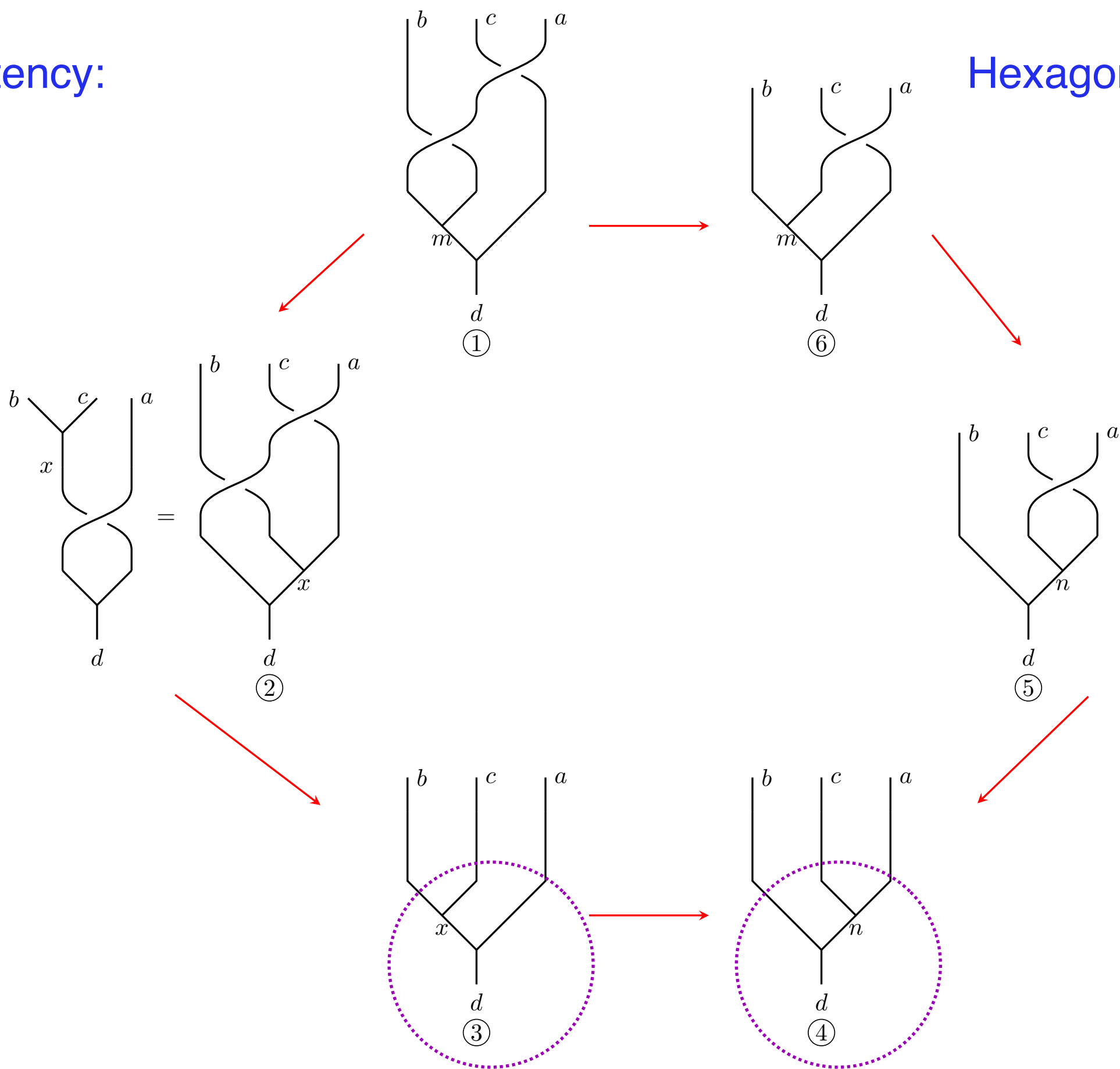
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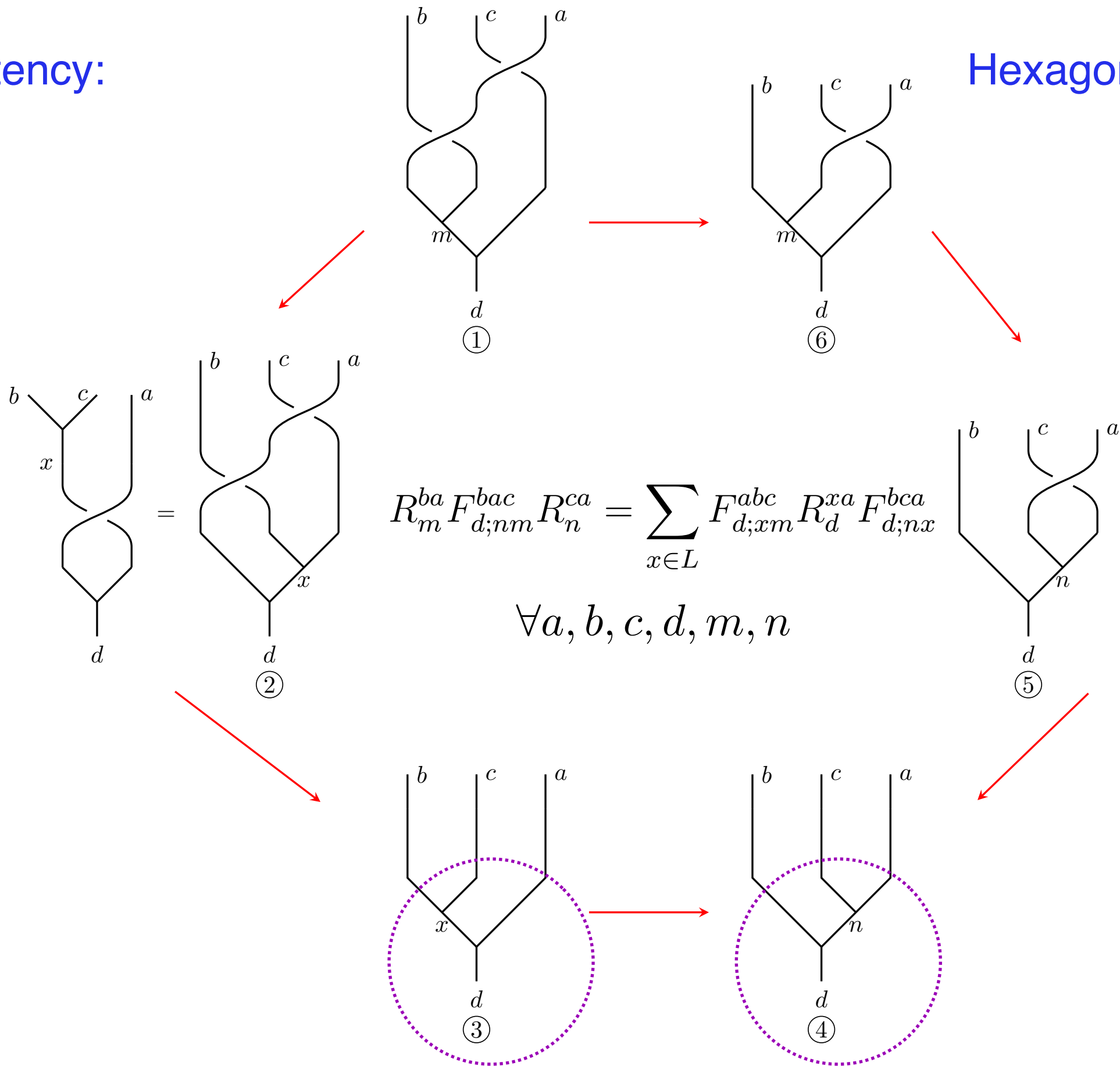
Consistency:

Hexagon Equation 1



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Hexagon Equation 1



Similarly, replacing braiding by the inverse braiding:

$$(R_m^{ab})^{-1} F_{d;nm}^{bac} (R_n^{ac})^{-1} = \sum_{x \in L} F_{d;xm}^{abc} (R_d^{ax})^{-1} F_{d;n x}^{bca}.$$

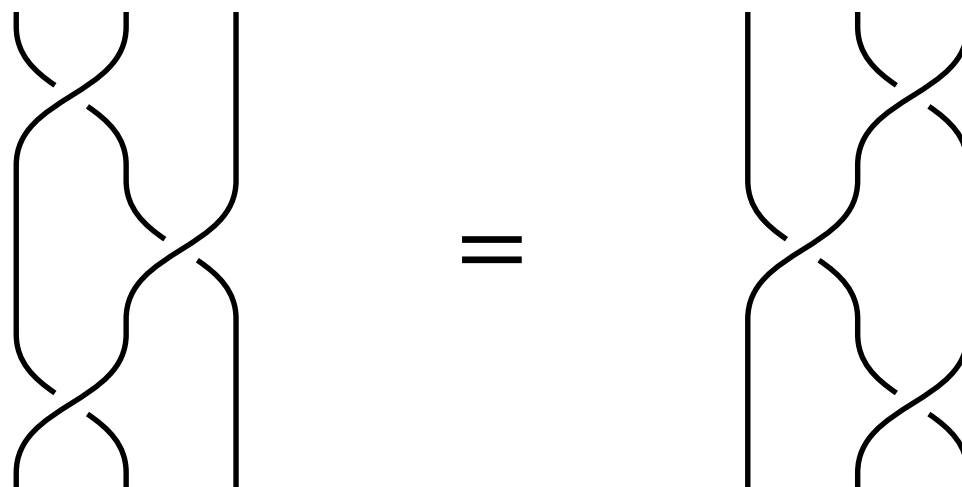
Hexagon Equation 2

Similarly, replacing braiding by the inverse braiding:

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Hexagon Equation 2

The Yang-Baxter Equation follows:

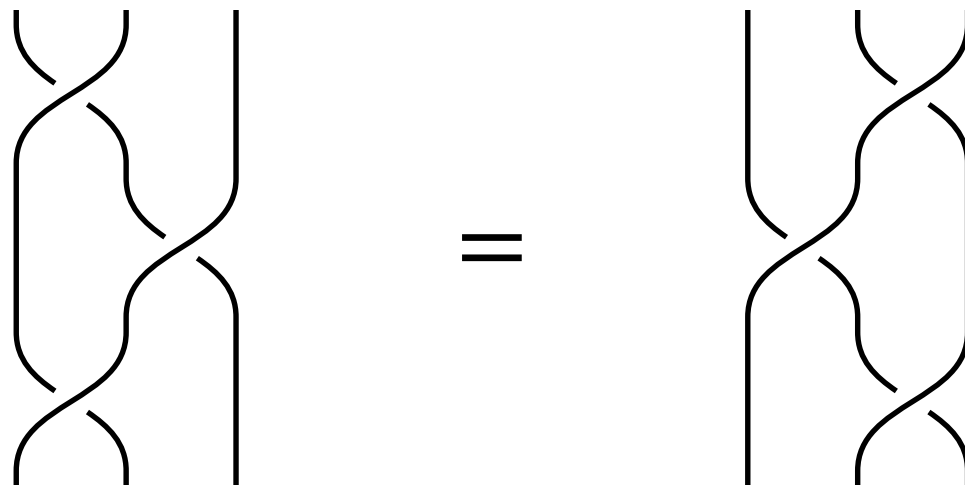


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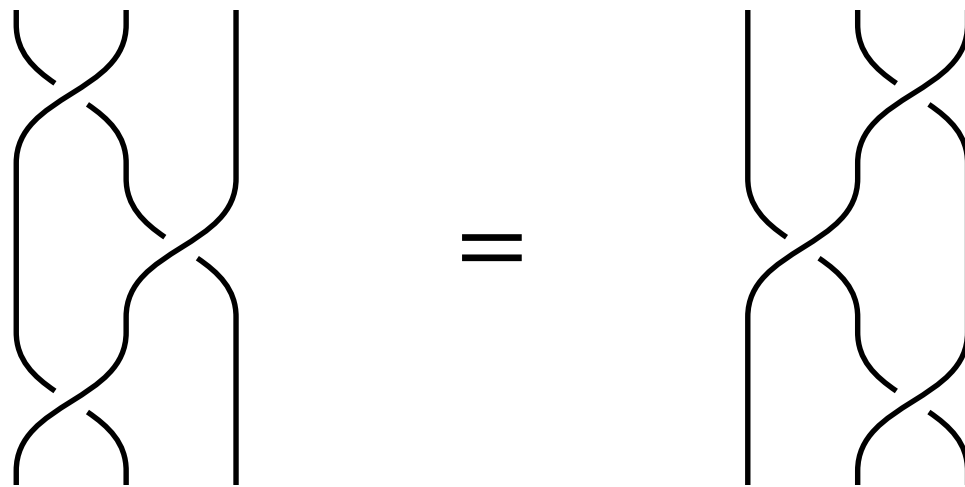
It also follows: $R_a^{a1} = R_a^{1a} = 1$

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It also follows: $R_a^{a1} = R_a^{1a} = 1$

Example: Fib

$$R_1^{\tau\tau} = e^{-4\pi i/5}, \quad R_\tau^{\tau\tau} = e^{3\pi i/5}$$

★ Summary

A (multiplicity-free) unitary fusion category (UFC) is a tuple:

$$\{L, N_{ab}^c, F_{d,ef}^{abc}\} \quad \text{satisfying}$$

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- F -symbols constraints:
 - Pentagon equations with unitary F -matrices
 - Compatibility with unit: $F_{c;ba}^{a1b} = 1$
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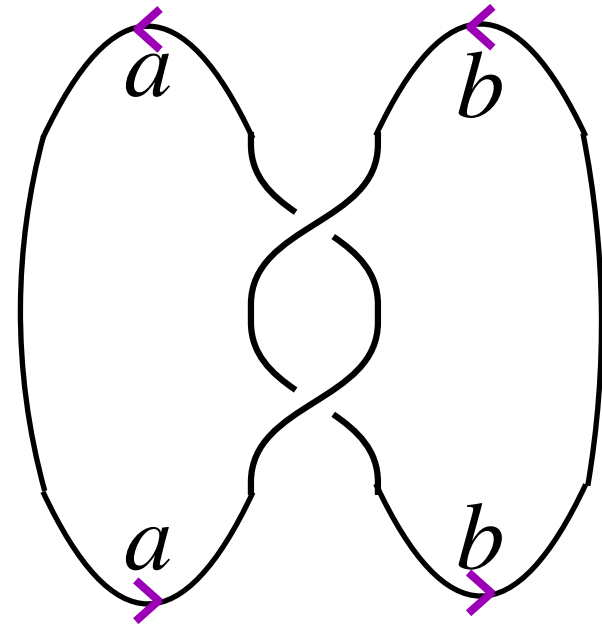
$$\{L, N_{ab}^c, F_{d,ef}^{abc}, R_c^{ba}\} \quad \text{satisfying}$$

- $\{L, N_{ab}^c, F_{d,ef}^{abc}\}$ UFC
- Hexagon equations with unitary R -symbols

A (multiplicity-free) unitary modular tensor category (UMTC) is a UBFC with a non-degenerate S-matrix:

$$S = (S_{ab})_{a,b \in L}$$

$$S_{ab} :=$$



A (multiplicity-free) anyon system is characterized by a UMTC

★ More examples

Ising

- Label set: $L = \{\mathbf{1}, \psi, \sigma\}$; $\bar{a} = a$.
- Fusion rules: $\psi \otimes \psi = \mathbf{1}$, $\psi \otimes \sigma = \sigma$, $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$.
- F -matrices:

$$F_{\sigma}^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F_{\sigma}^{\psi\sigma\psi} = F_{\psi}^{\sigma\psi\sigma} = (-1).$$

- R -symbols: $R_1^{\psi\psi} = -1$, $R_{\sigma}^{\psi\sigma} = R_{\sigma}^{\sigma\psi} = -i$, $R_1^{\sigma\sigma} = e^{-\pi i/8}$, $R_{\psi}^{\sigma\sigma} = e^{3\pi i/8}$.

Toric code

- Label set: $L = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\} \longleftrightarrow \{\mathbf{1}, e, m, f\}$; $\bar{a} = a$.
- Fusion rules: $a \otimes b = (a + b) \bmod 2$.
- F -matrices: (1) .
- R -symbols: $R_{a+b}^{a,b} = (-1)^{a_1 b_2}$, $a = (a_1, a_2), b = (b_1, b_2) \in L$.

★ Some related quantities: quantum dimension

$$\dim_q a \quad \equiv \quad |F_{a;\mathbf{1}\mathbf{1}}^{a\bar{a}a}|^{-1}$$

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Some facts:

- $\dim_q a = \dim_q \bar{a}$
 - If $a \otimes b = \bigoplus_c N_{ab}^c c$, then $\dim_q a \dim_q b = \sum_c N_{ab}^c \dim_q c$
- i.e. $\dim_q : \mathbb{Z}[L] \rightarrow \mathbb{R}$ is an algebra morphism invariant under bar.

★ Some related quantities: quantum dimension

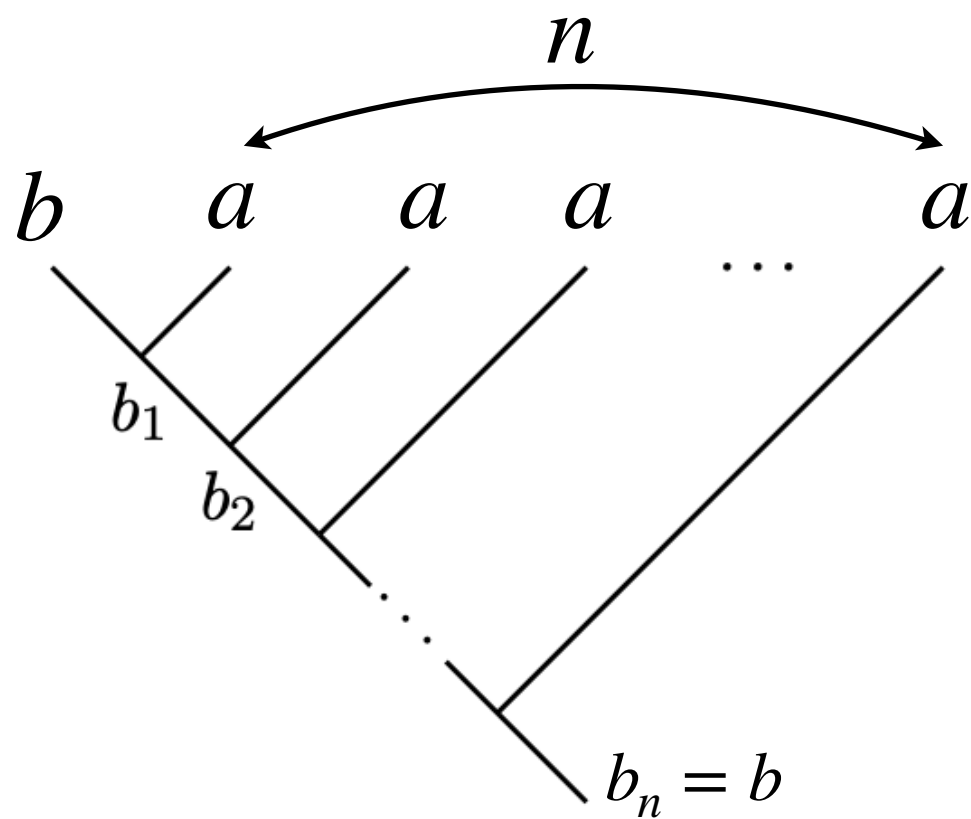
$$\dim_q a \quad \equiv \quad |F_{a;\mathbf{1}\mathbf{1}}^{a\bar{a}a}|^{-1}$$

Some facts:

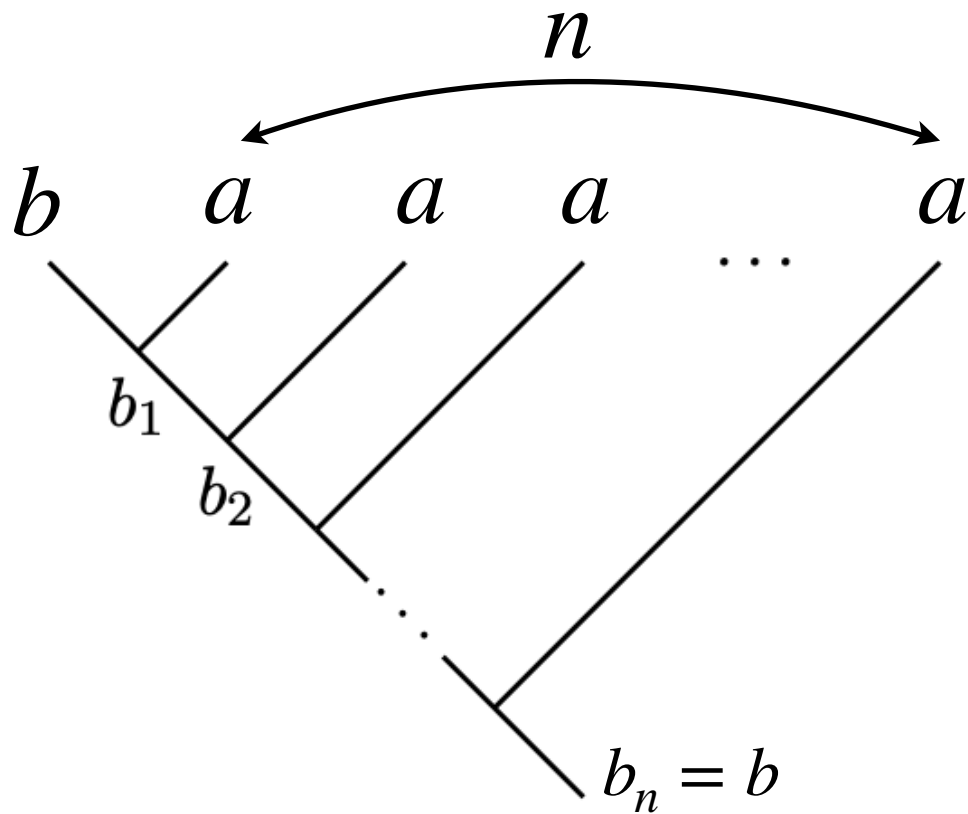
- $\dim_q a = \dim_q \bar{a}$
- If $a \otimes b = \bigoplus_c N_{ab}^c c$, then $\dim_q a \dim_q b = \sum_c N_{ab}^c \dim_q c$
i.e. $\dim_q : \mathbb{Z}[L] \rightarrow \mathbb{R}$ is an algebra morphism invariant under bar.
- $\dim_q a$ is the Frobenius-Perron dimension of a , that is, the largest eigenvalue, dominating in absolute value, of the matrix N_a , with $(N_a)_{bc} := N_{ab}^c$

Consider the space $V_b^{b,a,\dots,a}$

$$\dim V_b^{b,a,\dots,a} =$$



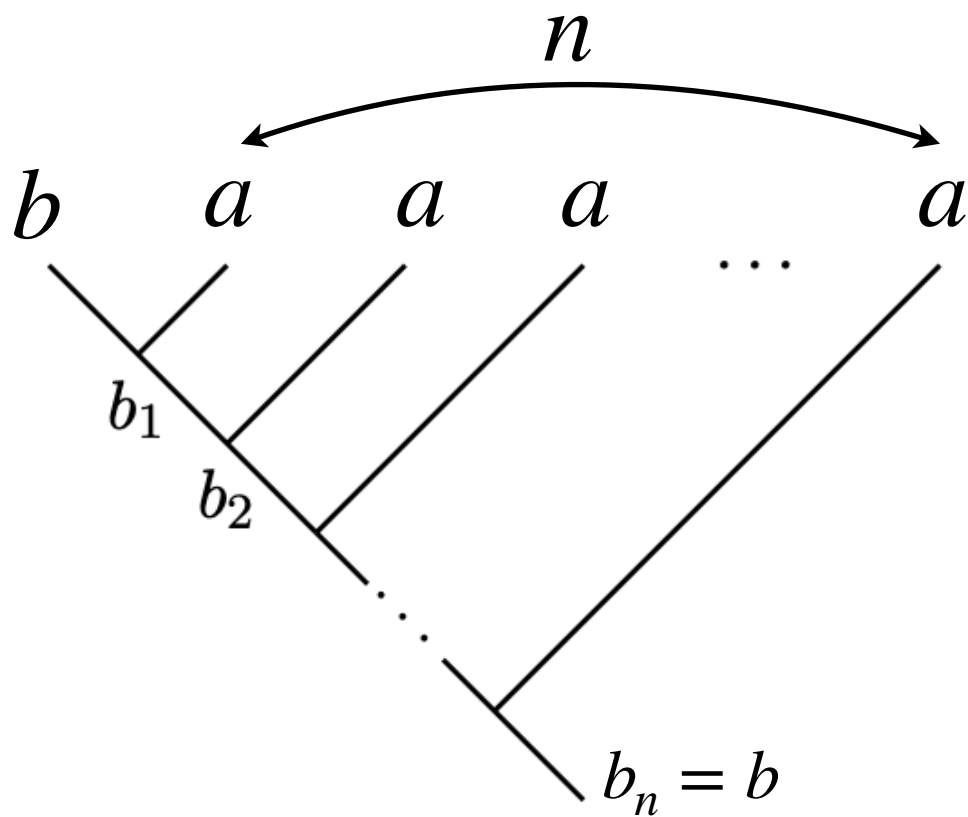
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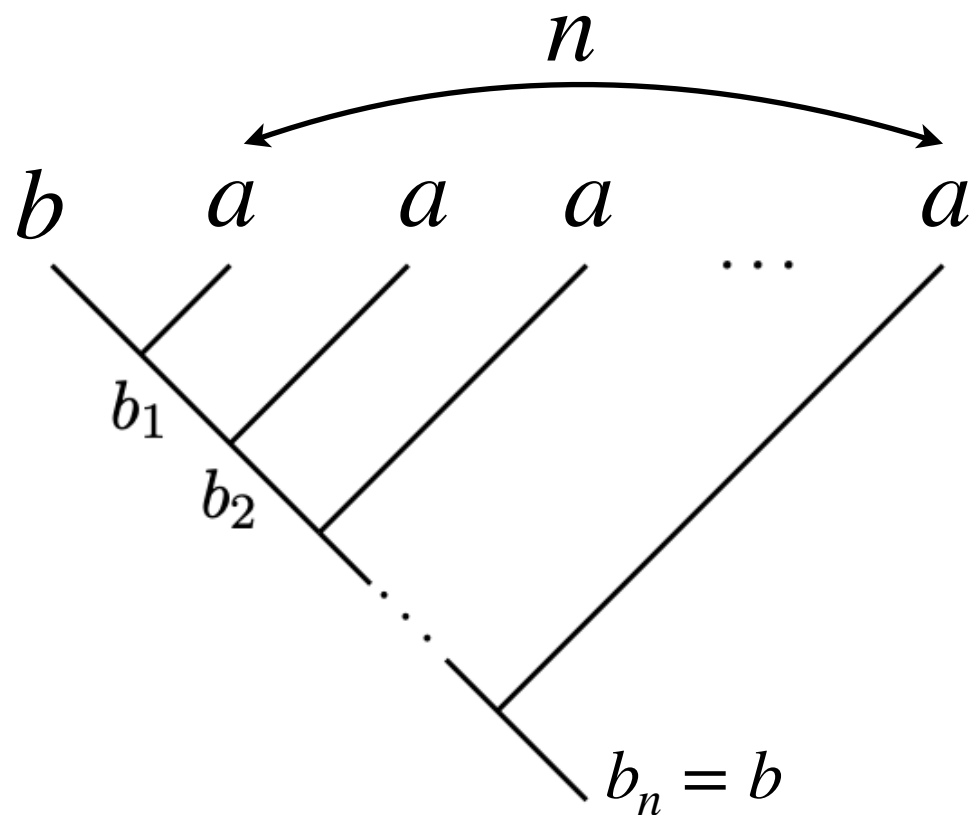


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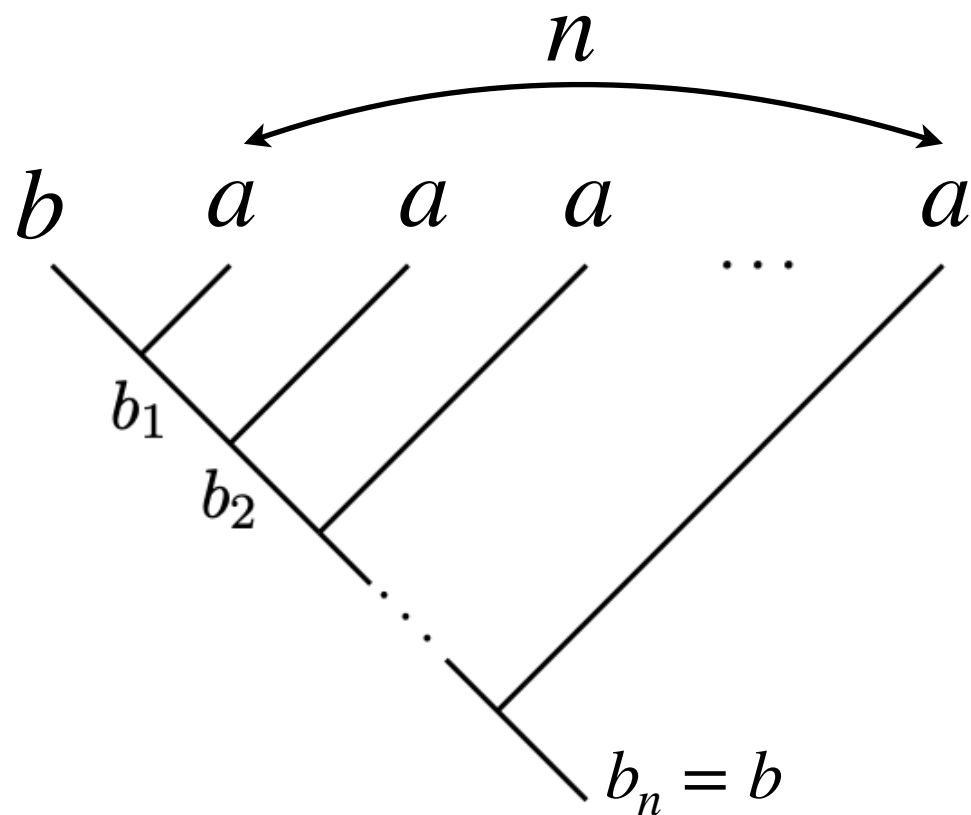
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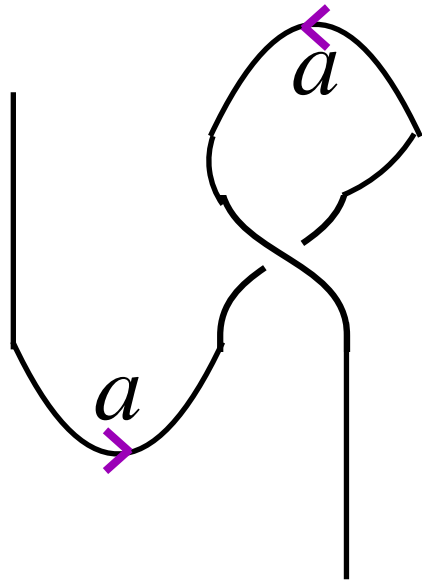
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Exercise: Fib

Fibonacci sequence $\{F_n\}: F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, F_1 = 1.$

$$\dim V_\tau^{\tau^{\otimes n}} = \dim V_1^{\tau^{\otimes(n+1)}} = F_n \qquad \dim_q \tau = \frac{1 + \sqrt{5}}{2}$$

★ Some related quantities: topological twist



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The diagram illustrates a topological twist through a sequence of three configurations:

- Left Configuration:** A vertical line on the left, a loop on the right, and a bottom arc. The bottom arc is labeled a with a purple arrow pointing right. The loop is labeled $\bar{a}$ with a purple arrow pointing left.
- Middle Configuration:** A graph with a vertical line on the left labeled a . It branches into a diagonal line labeled a and a curved line labeled $\bar{a}$. These meet at a vertex labeled 1 , which is connected to a vertical line on the right labeled a .
- Right Configuration:** A simplified version of the middle configuration, showing a vertical line on the left labeled a and a vertical line on the right labeled a , connected by a diagonal line labeled a and a curved line labeled $\bar{a}$.

★ Some related quantities: topological twist

Diagram 1: A vertical line with a loop labeled a and a purple arrow.

Diagram 2: A vertical line with a loop labeled a and a purple arrow, followed by a loop labeled $\bar{a}$ and a label 1 .

Diagram 3: A vertical line with a loop labeled a and a purple arrow, followed by a loop labeled $\bar{a}$ and a label 1 , and a coefficient $\dim_q(a) F_{a;11}^{a\bar{a}a}$.

★ Some related quantities: topological twist

$$\begin{array}{c}
 \text{Diagram 1: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.} \\
 \\
 \text{Diagram 2: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.} \\
 \\
 \text{Diagram 3: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.} \\
 \\
 \text{Diagram 4: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.}
 \end{array}
 = \dim_q(a) \begin{array}{c} \text{Diagram 5: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.} \end{array} = \dim_q(a) F_{a; \mathbf{11}}^{a\bar{a}a} \begin{array}{c} \text{Diagram 6: A vertical line on the left, a loop on the right with a purple arrow labeled } a \text{ at the top, and a purple arrow labeled } a \text{ at the bottom.} \end{array} = F_{a; \mathbf{11}}^{a\bar{a}a} \dim_q(a) \overline{R_1^{\bar{a}a}} \Big|_a$$

★ Some related quantities: topological twist

$$\begin{array}{c}
 \text{Diagram 1: A vertical line with a loop at the top. The loop has a purple arrow pointing left, labeled $\bar{a}$. The bottom of the loop has a purple arrow pointing right, labeled a .} \\
 \\
 = \dim_q(a) \text{ Diagram 2: A vertical line with a loop at the top. The loop has a purple arrow pointing left, labeled $\bar{a}$. The bottom of the loop has a purple arrow pointing right, labeled a .} = \\
 \\
 \dim_q(a) F_{a; \mathbf{11}}^{a\bar{a}a} \text{ Diagram 3: A vertical line with a loop at the top. The loop has a purple arrow pointing left, labeled $\bar{a}$. The bottom of the loop has a purple arrow pointing right, labeled a .} = \\
 \\
 F_{a; \mathbf{11}}^{a\bar{a}a} \dim_q(a) \overline{R_1^{\bar{a}a}} \Big|_a \\
 \text{topological twist } \theta_a
 \end{array}$$

★ Some related quantities: braid group representation

N-strand braid group

$$B_N = \langle \sigma_1, \dots, \sigma_N \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \mid i - j \mid \geq 2 \rangle$$

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Consider $V_b^{a^{\otimes n}} := V_b^{a, \dots, a}$. Braiding induces a unitary representation:

$$\rho_b^{a,n} : B_n \longrightarrow U(V_b^{a^{\otimes n}})$$

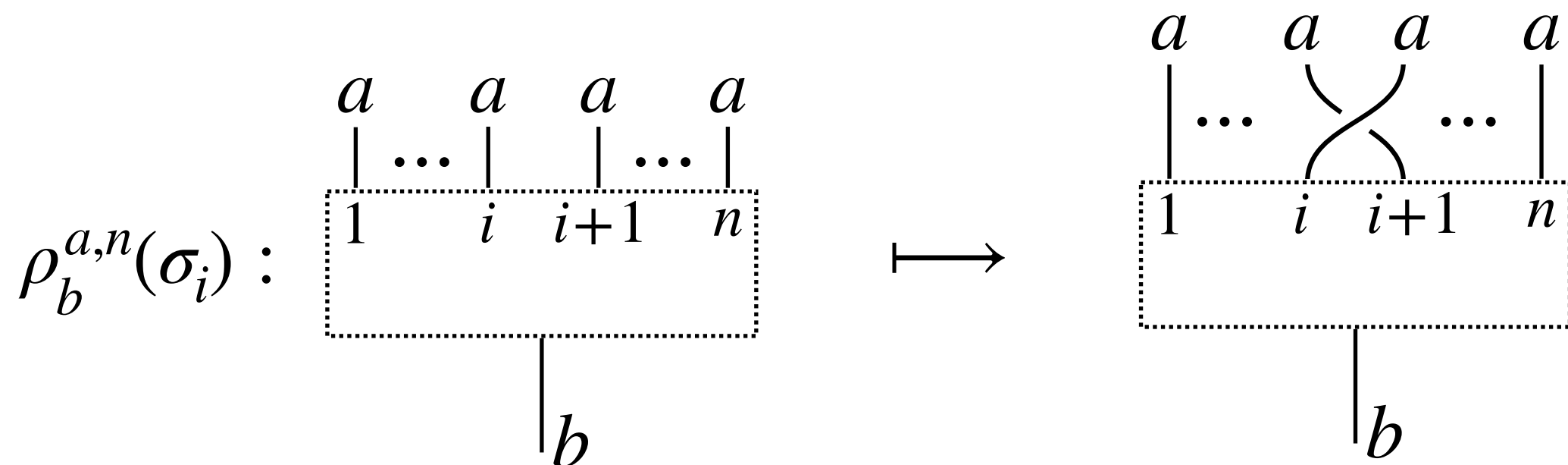
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- is endowed with a non-degenerate braiding

$$V \otimes W \simeq W \otimes V$$

UMTC in abstract category language:

anyon language	category language
anyon type	simple object
multiple anyons	tensor product $\otimes$
superposition of anyons	direct sum $\oplus$
state space $V_b^{a_1, \dots, a_n}$	Hom space $\text{Hom}(b, a_1 \otimes \dots \otimes a_n)$
physical processes	morphisms
anyon braiding	isomorphism: $a \otimes b \cong b \otimes a$
fusing rules	direct sum decomposition